DIFFERENTIAL EQUATION

1. DIFFERENTIAL EQUATION :

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**.

A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only e.g. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + \cos x = 0$ and it is said to be **partial** if there are two or more independent variables. e.g. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ is a partial differential equation. We are concerned with ordinary differential equations only.

2. ORDER OF DIFFERENTIAL EQUATION :

The order of a differential equation is the order of the highest differential coefficient occurring in it.

3. DEGREE OF DIFFERENTIAL EQUATION :

The exponent of the highest order differential coefficient, when the differential equation is expressed as a polynomial in all the differential coefficient.

Thus the differential equation :

$$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \dots\dots\dots = 0$$ is of order m & degree p.

Note :

(i) The exponents of all the differential coefficient should be free from radicals and fraction.

(ii) The degree is always positive natural number.

(iii) The degree of differential equation may or may not exist.

Illustration 1 : Find the order and degree of the following differential equation :

(i) $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx} + 3}$ (ii) $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$ (iii) $\frac{dy}{dx} = \sqrt{3x+5}$

Solution :

(i) The given differential equation can be re-written as $\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx} + 3\right)^2$

Hence order is 2 and degree is 3.

(ii) The given differential equation has the order 2. Since the given differential equation cannot be written as a polynomial in the differential coefficients, the degree of the equation is not defined.

(iii) Its order is 1 and degree 1. **Ans.**

Illustration 2 : The order and degree of the differential equation $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$ are -

(A) 2 , 2 (B) 2 , 3 (C) 3 , 2 (D) none of these

Solution : Clearly order is 2 and degree is 2 (from the definition of order and degree of differential equations).

Ans. (A)

Do yourself - 1 :

Find the order and degree of following differential equations

(i) $[1 + (y')^2]^{1/2} = x^2 + y$ **(ii)** $(1 + y')^{1/2} = y''$ **(iii)** $y' = \sin y$

4. FORMATION OF A DIFFERENTIAL EQUATION :

In order to obtain a differential equation whose solution is

$$f(x_1, y_1, c_1, c_2, c_3........, c_n) = 0$$

where $c_1, c_2, ...c_n$ are 'n' arbitrary constants, we have to eliminate the 'n' constants for which we require (n+1) equations.

A differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t the independent variable (say x) as many times as the number of independent arbitrary constants in it.

(b) Eliminate the arbitrary constants.

(c) The eliminant is the required differential equation.

Note :

(i) A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

(ii) For there being n differentiation, the resulting equation must contain a derivative of n^{th} order i.e. equal to number of independent arbitrary constant.

Illustration 3 : Find the differential equation of all parabolas whose axes is parallel to the x-axis and having latus rectum a.

Solution : Equation of parabola whose axes is parallel to x-axis and having latus rectum 'a' is $(y - \beta)^2 = a(x - \alpha)$

Differentiating both sides, we get $2(y - \beta)\dfrac{dy}{dx} = a$

Again differentiating, we get

$$\Rightarrow 2(y-\beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow a\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0.$$ **Ans.**

Illustration 4 : Find the differential equation whose solution represents the family : $c(y + c)^2 = x^3$

Solution : $c(y + c)^2 = x^3$ ...(i)

Differentiating, we get, $c.\left[2(y+c)\right]\dfrac{dy}{dx} = 3x^2$

Writing the value of c from (i), we have

$$\frac{2x^3}{(y+c)^2}(y + c)\frac{dy}{dx} = 3x^2 \quad \Rightarrow \quad \frac{2x^3}{y+c}\frac{dy}{dx} = 3x^2$$

i.e. $\frac{2x}{y+c}\frac{dy}{dx} = 3 \Rightarrow \frac{2x}{3}\left[\frac{dy}{dx}\right] = y + c$

Hence $c = \frac{2x}{3}\left[\frac{dy}{dx}\right] - y$

Substituting value of c in equation (i), we get $\left[\frac{2x}{3}\left(\frac{dy}{dx}\right) - y\right]\left[\frac{2x}{3}\frac{dy}{dx}\right]^2 = x^3$,

which is the required differential equation. **Ans.**

Illustration 5 : Find the differential equation whose solution represents the family : $y = a\cos\theta x + b\sin\theta x$, where θ = fixed constant

Solution : $y = a\cos\theta x + b\sin\theta x$, θ = fixed constant ....(i)

Differentiating, we get $\frac{dy}{dx} = -\theta a\sin\theta x + \theta b\cos\theta x$

Again differentiating, we get $\frac{d^2y}{dx^2} = -\theta^2 a\cos\theta x - \theta^2 b\sin\theta x$

using equation (i), we get $\frac{d^2y}{dx^2} = -\theta^2 y$ **Ans.**

Do yourself - 2

Eliminate the arbitrary constants and obtain the differential equation satisfied by it.

(i) $y = 2x + ce^x$ (ii) $y = \left(\frac{a}{x^2}\right) + bx$ (iii) $y = ae^{2x} + be^{-2x} + c$

5. SOLUTION OF DIFFERENTIAL EQUATION :

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e., from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx} = e^x$ could be obtained by simply integrating both sides, i.e., $y = e^x + c$ and that of, $\frac{dy}{dx} = px + q$ is $y = \frac{px^2}{2} + qx + c$, where c is arbitrary constant.

(i) **A general solution** or an integral of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation.

For example, a general solution of the differential equation $\frac{d^2x}{dt^2} = -4x$ is $x = A\cos 2t + B\sin 2t$ where A and B are the arbitrary constants.

(ii) **Particular solution or particular integral** is that solution of the differential equation which is obtained from the general solution by assigning particular values to the arbitrary constant in the general solution. For example, $x = 10\cos 2t + 5\sin 2t$ is a particular solution of differential equation $\frac{d^2x}{dt^2} = -4x$.

Note :

(i) The general solution of a differential equation can be expressed in different (but equivalent) forms. For example

$$\log x - \log (y + 2) = k \qquad \text{....(i)}$$

where k is an arbitrary constant is the general solution of the differential equation $xy' = y + 2$. The solution given by equation (i) can also be re-written as

$$\log\left(\frac{x}{y+2}\right) = k \text{ or } \frac{x}{y+2} = e^k = c_1 \qquad \text{...(ii)}$$

or $\quad x = c_1 (y + 2) \qquad$ ...(iii)

where $c_1 = e^k$ is another arbitrary constant. The solution (iii) can also be written as

$$y + 2 = c_2 x$$

where $c_2 = 1/c_1$ is another arbitrary constant.

(ii) All differential equations that we come across have unique solutions or a family of solutions. For example, the differential equation $\left|\frac{dy}{dx}\right| + |y| = 0$ has only the trivial solution, i.e. $y = 0$.

The differential equation $\left|\frac{dy}{dx}\right| + |y| + c = 0,\ c > 0$ has no solution.

6. ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS :

(a) Separation of Variables :

Some differential equations can be solved by the method of separation of variables (or "variable separable"). This method is only possible, if we can express the differential equation in the form

$$A(x)dx + B(y)\,dy = 0$$

where A(x) is a function of 'x' only and B(y) is a function of 'y' only.

A general solution of this is given by,

$$\int A(x)\,dx + \int B(y)dy = c$$

where 'c' is the arbitrary constant.

Illustration 6 : Solve the differential equation $xy\,\frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1 + x + x^2)$.

Solution : Differential equation can be rewritten as

$$xy\,\frac{dy}{dx} = (1 + y^2)\left(1 + \frac{x}{1+x^2}\right) \Rightarrow \frac{y}{1+y^2}\,dy = \left(\frac{1}{x} + \frac{1}{1+x^2}\right)dx$$

Integrating, we get

$$\frac{1}{2}\ell n(1 + y^2) = \ell n\ x + \tan^{-1} x + \ell n\ c \Rightarrow \sqrt{1+y^2} = cxe^{\tan^{-1} x}.$$ **Ans.**

Illustration 7 : Solve the differential equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$.

Solution : The given equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

$$\Rightarrow \frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0 \qquad \Rightarrow \int\left(\frac{1}{y^3} - \frac{1}{y}\right)dy + \int\left(\frac{1}{x^3} + \frac{1}{x}\right)dx = 0$$

$$\log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$$ **Ans.**

Overlooked solution :

Illustration 8 : Solve : $\frac{dy}{dx} = (x - 3)(y + 1)^{2/3}$

Solution : $\frac{dy}{dx} = (x-3)(y+1)^{2/3}$

$$\int \frac{dy}{(y+1)^{2/3}} = \int (x-3)dx$$

Integrate and solve for y : $3(y + 1)^{1/3} = \frac{1}{2}(x - 3)^2 + C$

$(y + 1)^{1/3} = \frac{1}{6}(x - 3)^2 + C_0 \Rightarrow y+1 = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 \Rightarrow y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$

All of this looks routine. However, **note that y = –1 is a solution to the original equation** $\frac{dy}{dx} = 0$ and $(x - 3)(y + 1)^{2/3} = 0$

However, we can not obtain $y = -1$ from $y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$ by setting constant C_0 equal to any number. (We need to find a constant which makes $\frac{1}{6}(x - 3)^2 + C_0 = 0$ for all x.)

Two points emerge from this.

(i) We may sometime miss solutions while performing certain algebraic operations (in this case, division).

(ii) We don't always get every solution to a differential equation by assigning values to the arbitrary constants.

Do yourself - 3 :

Solve the following differential equations :

(i) $\frac{2dy}{dx} = \frac{y(x+1)}{x}$ (ii) $\sqrt{1+4x^2}\,dy = y^3 x\,dx$

(iii) $(\tan y)\frac{dy}{dx} = \sin(x + y) + \sin(x - y)$

(i) **Equation of the form :**

$\Rightarrow$ **$y' = f(ax + by + c), b \neq 0$**

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Illustration 9 : Solve $\frac{dy}{dx} = \cos(x + y) - \sin(x + y)$.

Solution : $\frac{dy}{dx} = \cos(x + y) - \sin(x + y)$

Substituting, $x + y = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore $\frac{dt}{dx} - 1 = \cos t - \sin t$

$$\Rightarrow \int \frac{dt}{1+\cos t - \sin t} = \int dx \Rightarrow \int \frac{\sec^2 \frac{t}{2}\,dt}{2\left(1 - \tan\frac{t}{2}\right)} = \int dx \Rightarrow -\ell n\left|1 - \tan\frac{x+y}{2}\right| = x + c.$$ **Ans.**

Illustration 10: Solve : $y' = (x + y + 1)^2$

Solution : $y' = (x + y + 1)^2$ ....(i)

Let $t = x + y + 1$

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting in equation (i) we get

$$\frac{dt}{dx} = t^2 + 1 \Rightarrow \int \frac{dt}{1+t^2} = \int dx \Rightarrow \tan^{-1} t = x + C \Rightarrow t = \tan(x + C)$$

$x + y + 1 = \tan(x + C) \Rightarrow y = \tan(x + C) - x - 1$ **Ans.**

Do yourself - 4 :

Solve the following differential equations :

(i) $\frac{dy}{dx} = (y - 4x)^2$ (ii) $\tan^2(x + y)dx - dy = 0$

(ii) Equation of the form : $\Rightarrow \frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

Case I : If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ then

Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$ then $a_1 = \lambda a_2$ .....(i) ; $b_1 = \lambda b_2$ ......(ii)

from (i) and (ii), differential equation becomes

$$\frac{dy}{dx} = \frac{\lambda a_2x + \lambda b_2y + c_1}{a_2x + b_2y + c_2} \Rightarrow \frac{dy}{dx} = \frac{\lambda(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2}$$

or we can say, $\frac{dy}{dx} = f(a_2x + b_2y)$

which can be solved by substituting $t = a_2x + b_2y$

Illustration 11 : Solve : $(x + y)dx + (3x + 3y - 4) dy = 0$

Solution : Let $t = x + y$

$\Rightarrow dy = dt - dx$

So we get, $tdx + (3t - 4)(dt - dx) = 0$

$$2dx + \left(\frac{3t-4}{2-t}\right)dt = 0 \Rightarrow 2dx - 3dt + \frac{2}{2-t}dt = 0$$

Integrating and replacing t by x + y, we get

$2x - 3t - 2[\ell n|(2 - t)|] = c_1$

$\Rightarrow 2x - 3(x + y) - 2[\ell n|(2 - x - y)|] = c_1$

$\Rightarrow x + 3y + 2\ell n|(2 - x - y)| = c$ **Ans.**

Case II : **If $a_2 + b_1 = 0$, then** a simple cross multiplication and substituting d(xy) for xdy + ydx and integrating term by term, yield the results easily.

Illustration 12 : Solve $\frac{dy}{dx} = \frac{x - 2y + 1}{2x + 2y + 3}$

Solution : $\frac{dy}{dx} = \frac{x - 2y + 1}{2x + 2y + 3}$

$\Rightarrow 2xdy + 2y\,dy + 3dy = xdx - 2y\,dx + dx$

$\Rightarrow (2y + 3)\,dy = (x + 1)\,dx - 2(xdy + ydx)$

On integrating, we get

$\Rightarrow \int (2y+3)dy = \int (x+1)dx - \int 2d(xy)$

Solving : $y^2 + 3y = \dfrac{x^2}{2} + x - 2xy + c$ **Ans.**

Do yourself - 5 :

Solve the following differential equations :

(i) $\dfrac{dy}{dx} = \dfrac{2x-y+2}{2y-4x+1}$ (ii) $\dfrac{dy}{dx} = \dfrac{3x-5y}{5x+y+3}$

(iii) Equation of the form :

$\Rightarrow$ **yf(xy)dx + xg(xy)dy = 0** .... (i)

The substitution xy = z, reduces differential equation of this form to the form in which the variables are separable.

Let xy = z .... (ii)

$dy = \left[\dfrac{xdz - zdx}{x^2}\right]$ .... (iii)

using equation (ii) & (iii), equation (i) becomes $\dfrac{z}{x}f(z)dx + xg(z)\left[\dfrac{xdz - zdx}{x^2}\right] = 0$

$\Rightarrow \dfrac{z}{x}f(z)dx + g(z)dz - \dfrac{z}{x}g(z)dx = 0 \Rightarrow \dfrac{z}{x}\{f(z)-g(z)\}dx + g(z)dz = 0 \Rightarrow \dfrac{1}{x}dx + \dfrac{g(z)dz}{z\{f(z)-g(z)\}} = 0$

Illustration 13 : Solve $y(xy+1)dx + x(1+xy+x^2y^2)dy = 0$

Solution : Let xy = v

$y = \dfrac{v}{x} \Rightarrow dy = \dfrac{xdv - vdx}{x^2}$

Now, differential equation becomes $\Rightarrow \dfrac{v}{x}(v+1)dx + x(1+v+v^2)\left(\dfrac{xdv - vdx}{x^2}\right) = 0$

On solving, we get

$v^3dx - x(1+v+v^2)\,dv = 0$

separating the variables & integrating we get

$\Rightarrow \int \dfrac{dx}{x} - \int\left(\dfrac{1}{v^3} + \dfrac{1}{v^2} + \dfrac{1}{v}\right)dv = 0 \Rightarrow \ell nx + \dfrac{1}{2v^2} + \dfrac{1}{v} - \ell nv = c$

$\Rightarrow 2v^2\ell n\left(\dfrac{v}{x}\right) - 2v - 1 = -2cv^2 \Rightarrow 2x^2y^2\ell ny - 2xy - 1 = Kx^2y^2$ where $K = -2c$

Do yourself - 6 :

Solve the following differential equations :

(i) $(y - xy^2)dx - (x + x^2y)dy = 0$ (ii) $y(1+2xy)dx + x(1-xy)dy = 0$

(iv) Transformation to polar-co-ordinates :

Sometimes conversion of cartesian co-ordinates into polar coordinates helps us in separating the variables.

(1) $x = r\cos\theta,\quad y = r\sin\theta$
then $x^2 + y^2 = r^2$
$xdx + ydy = rdr$
$xdy - ydx = r^2 d\theta$

(2) $x = r\sec\theta,\quad y = r\tan\theta$
then $x^2 - y^2 = r^2$
$xdx - ydy = rdr$
$xdy - ydx = r^2 \sec\theta\, d\theta$

Illustration 14 : Solve : $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$

Solution : The given equation can be reduced to

$$\frac{xdx + ydy}{ydx - xdy} = \frac{(x^2+y^2)^2}{x^2}$$

Substituting $x = r\cos\theta$
$y = r\sin\theta$

we get, $\dfrac{rdr}{r^2 d\theta} = \dfrac{-(r^2)^2}{r^2\cos^2\theta} \Rightarrow \displaystyle\int\frac{dr}{r^3} = -\int\sec^2\theta d\theta \Rightarrow -\frac{1}{2r^2} = -\tan\theta + c$

Substituting, $\dfrac{1}{2(x^2+y^2)} = \dfrac{y}{x} + K$ **Ans.**

Do yourself - 7 :

Solve the following differential equations :

(i) $xdx + ydy = xdy - ydx$ (ii) $ydx - xdy = xy\,dy - x^2dx$

(b) **Homogeneous equations :**

A function $f(x,y)$ is said to be a homogeneous function of degree n, if the substitution $x = \lambda x$, $y = \lambda y$, $\lambda > 0$ produces the equality

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$

The degree of homogeneity 'n' can be any real number.

Illustration 15 : Find the degree of homogeneity of function

(i) $f(x,y) = x^2 + y^2$ (ii) $f(x,y) = (x^{3/2} + y^{3/2})/(x + y)$

(iii) $f(x,y) = \sin\left(\dfrac{x}{y}\right)$

Solution :

(i) $f(\lambda x, \lambda y) = \lambda^2x^2 + \lambda^2y^2$
$= \lambda^2(x^2 + y^2)$
$= \lambda^2 f(x,y)$
degree of homogeneity $\rightarrow$ 2

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^{3/2}x^{3/2} + \lambda^{3/2}y^{3/2}}{\lambda x + \lambda y}$
$f(\lambda x, \lambda y) = \lambda^{1/2} f(x,y)$
degree of homogeneity $\rightarrow$ 1/2

(iii) $f(\lambda x, \lambda y) = \sin\left(\dfrac{\lambda x}{\lambda y}\right) = \lambda^{\circ}\sin\left(\dfrac{x}{y}\right) = \lambda\ f(x,y)$
degree of homogeneity $\rightarrow$ 0

Illustration 16 : Determine whether or not each of the following functions is homogeneous.

(i) $f(x,y) = x^2 - xy$ (ii) $f(x,y) = \frac{xy}{x+y^2}$ (iii) $f(x,y) = \sin xy$

Solution : (i) $f(\lambda x, \lambda y) = \lambda^2x^2 - \lambda^2 xy$

$= \lambda^2(x^2 - xy) = \lambda^2 f(x,y)$ homogeneous.

(ii) $f(\lambda x,\lambda y) = \frac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x,y)$ not homogeneous.

(iii) $f(\lambda x, \lambda y) = \sin(\lambda^2 xy) \neq \lambda^n f(x,y)$ not homogeneous.

Do yourself - 8 :

(i) Find the degree of homogeneity of function $f(x,y) = x^3 \ell n\left[\sqrt{x+y}/\sqrt{x-y}\right]$

(ii) Find the degree of homogeneity of function $f(x,y) = ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3}$

(iii) Determine whether or not each of the following functions is homogeneous.

(a) $f(x,y) = \sqrt{x^2+2xy+3y^2}$ (b) $f(x,y) = x + y\cos\frac{y}{x}$ (c) $f(x,y) = x\sin y + y \sin x$.

(i) **Homogeneous first order differential equation**

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where $f(x,y)$ and $g(x,y)$ are homogeneous functions of x,y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting

$$y = vx,$$

so that the dependent variable y is changed to another variable v.

Since $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree say, n, they can be written as

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right) \quad \text{and} \quad g(x,y) = x^n g_1\left(\frac{y}{x}\right).$$

As $y = vx$, we have $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

The given differential equation, therefore, becomes

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{g_1(v)}$$

$$\Rightarrow \frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

so that the variables v and x are now separable.

Note : Sometimes homogeneous equation can be solved by substituting $x = vy$ or by using polar coordinate substitution.

Illustration 17 : The solution of the differential equation $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$ is -

(A) $\sin^2 y = x\sin y + \frac{x^2}{2} + c$ (B) $\sin^2 y = x\sin y - \frac{x^2}{2} + c$

(C) $\sin^2 y = x + \sin y + \frac{x^2}{2} + c$ (D) $\sin^2 y = x - \sin y + \frac{x^2}{2} + c$

Solution : Here, $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$

$\Rightarrow \quad \cos y \frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x}$, (put sin y = t)

$\Rightarrow \quad \frac{dt}{dx} = \frac{t + x}{2t - x}$ (put t = vx)

$\frac{xdv}{dx} + v = \frac{vx + x}{2vx - x} = \frac{v+1}{2v-1}$

$\therefore \quad x\frac{dv}{dx} = \frac{v+1}{2v-1} - v = \frac{v+1-2v^2+v}{2v-1}$

or $\frac{2v-1}{-2v^2+2v+1}dv = \frac{dx}{x}$ on solving, we get

$\sin^2 y = x \sin y + \frac{x^2}{2} + c$. **Ans. (A)**

Illustration 18 : Solve the differential equation $(1 + 2e^{x/y})\,dx + 2e^{x/y}(1 - x/y)\,dy = 0$.

Solution : The equation is homogeneous of degree 0.

Put x = vy, dx = v dy + y dv,

Then, differential equation becomes

$(1 + 2e^v)(v\,dy + y\,dv) + 2e^v(1 - v)\,dy = 0 \quad \Rightarrow (v + 2e^v)\,dy + y(1 + 2e^v)\,dv = 0$

$\frac{dy}{y} + \frac{1+2e^v}{v+2e^v}dv = 0$

Integrating and replacing v by x/y, we get

$\ell n\, y + \ell n(v + 2e^v) = \ell n c$ and $x + 2ye^{x/y} = c$ **Ans.**

Do yourself - 9 :

Solve the following differential equations :

(i) $y' = \frac{3x - y}{x + y}$ (ii) $(x - y\ell ny + y\ell nx)\,dx + x(\ell ny - \ell nx)\,dy = 0$

(iii) $(3xy + y^2)dx + (x^2 + xy)dy = 0,\ y(1) = 1$

(ii) Equations reducible to homgeneous form

The equation of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

can be reduced to homgeneous form by changing the variable x, y to u,v as

x = u + h , y = v + k

where h,k are the constants to be chosen so as to make the given equation homgeneous. We have

$\frac{dy}{dx} = \frac{dv}{du}$

$\therefore$ The equation becomes, $\frac{dv}{du} = \frac{a_1u + b_1v + (a_1h + b_1k + c_1)}{a_2u + b_2v + (a_2h + b_2k + c_2)}$

Let h and k be chosen so as to satisfy the equation

$a_1h + b_1k + c_1 = 0$ ...(i)

$a_2h + b_2k + c_2 = 0$ ...(ii)

Solve for h and k from (i) and (ii)

Now $\frac{du}{dv} = \frac{a_1u + b_1v}{a_2u + b_2v}$

is a homgeneous equation and can be solved by substituting v = ut.

Illustration 19 : Solve $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

Solution : Put x = X + h, y = Y + k

We have $\frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)}$

To determine h and k, we write

$h + 2k + 3 = 0,\ 2h + 3k + 4 = 0 \Rightarrow h = 1,\ k = -2$

So that $\frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$

Putting Y = VX, we get

$V + X\frac{dV}{dX} = \frac{1+2V}{2+3V} \Rightarrow \frac{2+3V}{3V^2-1}dV = -\frac{dX}{X}$

$\Rightarrow \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)} - \frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right]dV = -\frac{dX}{X}$

$\Rightarrow \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V - 1) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V + 1) = (-\log X + c)$

$\frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y - X) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y + X) = A$ where A is another constant and X = x − 1, Y = y + 2. **Ans.**

Do yourself - 10 :

(i) Solve the differential equation : $\frac{dy}{dx} = \frac{x+2y-5}{2x+y-4}$

(c) Linear differential equations :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together.

The n^{th} order linear differential equation is of the form ;

$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots + a_n(x)y = \phi(x)$, where $a_0(x)$, $a_1(x)$ .... $a_n(x)$ are called the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear . e.g. the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

Illustration 20: Which of the following equation is linear ?

(A) $\frac{dy}{dx} + xy^2 = 1$ (B) $x^2 \frac{dy}{dx} + y = e^x$ (C) $\frac{dy}{dx} + 3y = xy^2$ (D) $x\frac{dy}{dx} + y^2 = \sin x$

Solution : Clearly answer is (B)

Illustration 21 : Which of the following equation is non-linear ?

(A) $\frac{dy}{dx} = \cos x$ (B) $\frac{d^2y}{dx^2} + y = 0$ (C) $dx + dy = 0$ (D) $x\left(\frac{dy}{dx}\right)+\frac{3}{\left(\frac{dy}{dx}\right)} = y^2$

Solution : Clearly answer is (D)

(i) Linear differential equations of first order :

The most general form of a linear differential equation of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int Pdx}$.

So that we get $e^{\int Pdx}\left[\frac{dy}{dx} + Py\right] = Qe^{\int Pdx}$ ....(i)

$$\frac{d}{dx}\left(e^{\int Pdx}.y\right) = Qe^{\int Pdx} \quad ...(ii)$$

On integrating equation (ii), we get

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$$

This is the required general solution.

Note :

(i) The factor $e^{\int Pdx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I.F.

(ii) Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ; $(x + y + 1)\frac{dy}{dx} = y^2 + 3$ can be written as $(y^2 + 3)\frac{dx}{dy} = x + y + 1$ which is a linear differential equation.

Illustration 22 : Solve $(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$.

Solution : Differential equation can be rewritten as $(1 + y^2)\frac{dx}{dy} + x = e^{\tan^{-1}y}$

or $\frac{dx}{dy} + \frac{1}{1+y^2}.x = \frac{e^{\tan^{-1}y}}{1+y^2}$ ...(i)

I. F $= e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

so solution is $xe^{\tan^{-1}y} = \int\frac{e^{\tan^{-1}y}e^{\tan^{-1}y}}{1+y^2}dy$

Let $e^{\tan^{-1} y} = t \Rightarrow \dfrac{e^{\tan^{-1} y}}{1+y^2}\,dy = dt$

$xe^{\tan^{-1} y} = \int t\,dt$ [Putting $e^{\tan^{-1} y} = t$]

or $x\,e^{\tan^{-1} y} = \dfrac{t^2}{2} + \dfrac{c}{2} \Rightarrow 2x\,e^{\tan^{-1} y} = e^{2\tan^{-1} y} + c.$ **Ans.**

Illustration 23 : The solution of differential equation $(x^2 - 1)\dfrac{dy}{dx} + 2xy = \dfrac{1}{x^2-1}$ is -

(A) $y(x^2-1) = \dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right| + C$

(B) $y(x^2+1) = \dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right| - C$

(C) $y(x^2-1) = \dfrac{5}{2}\log\left|\dfrac{x-1}{x+1}\right| + C$

(D) none of these

Solution : The given differential equation is

$$(x^2-1)\frac{dy}{dx} + 2\ xy = \frac{1}{x^2-1} \quad \Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{(x^2-1)^2} \qquad \ldots(i)$$

This is linear differential equation of the form

$\dfrac{dy}{dx} + Py = Q$, where $P = \dfrac{2x}{x^2-1}$ and $Q = \dfrac{1}{(x^2-1)^2}$

$$\therefore\ \text{I.F.} = e^{\int P\,dx} = e^{\int 2x/(x^2-1)dx} = e^{\log(x^2-1)} = (x^2-1)$$

multiplying both sides of (i) by I.F. = $(x^2 - 1)$, we get

$$(x^2-1)\frac{dy}{dx} + 2\ xy = \frac{1}{x^2-1}$$

integrating both sides we get

$y(x^2-1) = \int \dfrac{1}{x^2-1}dx + C$ [Using : $y\ (\text{I.F.}) = \int Q.(\text{I.F.})\ dx + C$]

$$\Rightarrow\ y\,(x^2-1) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.$$

This is the required solution. **Ans.(A)**

Do yourself - 11 :

Solve the following differential equations :

(i) $\dfrac{xdy}{dx} = 2y + x^4 + 6x^2 + 2x,\ x \neq 0$

(ii) $(x-a)\dfrac{dy}{dx} + 3y = 12(x-a)^3,\ x > a > 0$

(iii) $y\ \ell n\ y\ dx + (x - \ell n\ y)\ dy = 0$

(ii) Equation reducible to linear form :

The equation of the form $\dfrac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x,

is called **Bernoulli's equation.**

On dividing by y^n, we get $y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q$

Let $y^{-n+1} = t$, so that $(-n+1)y^{-n}\frac{dy}{dx}=\frac{dt}{dx}$

then equation becomes $\frac{dt}{dx} + P(1-n)t = Q(1-n)$

which is linear with t as a dependent variable.

Illustration 24: Solve the differential equation $x\frac{dy}{dx}+y=x^3y^6$.

Solution : The given differential equation can be written as

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{xy^5}=x^2$$

Putting $y^{-5} = v$ so that

$-5\,y^{-6}\frac{dy}{dx}=\frac{dv}{dx}$ or $y^{-6}\frac{dy}{dx}=-\frac{1}{5}\frac{dv}{dx}$ we get

$$-\frac{1}{5}\frac{dv}{dx}+\frac{1}{x}v=x^2 \Rightarrow \frac{dv}{dx}-\frac{5}{x}v=-5x^2 \qquad \text{......(i)}$$

This is the standard form of the linear deferential equation having integrating factor

$$\text{I.F} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

Multiplying both sides of (i) by I.F. and integrating w.r.t. x

We get $v\,.\,\frac{1}{x^5}=\int -5x^2\,.\,\frac{1}{x^5}dx$

$$\Rightarrow \frac{v}{x^5}=\frac{5}{2}x^{-2}+c$$

$\Rightarrow y^{-5}x^{-5}=\frac{5}{2}x^{-2}+c$ which is the required solution. **Ans.**

Illustration 25 : Find the solution of differential equation $\frac{dy}{dx}-y\tan x=-y^2\sec x$.

Solution : $\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}\tan x=-\sec x$

$\frac{1}{y}=v$; $\frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$ $\quad\therefore \frac{-dv}{dx}-v\tan x=-\sec x$

$\frac{dv}{dx}+v\tan x=\sec x$, $\quad$ Here P = tan x, Q = sec x

I.F. = $e^{\int \tan x dx}=|\sec x|$ $\quad$ $v\,|\sec x| = \int \sec^2 x\,dx + c$

Hence the solution is $y^{-1}\,|\sec x| = \tan x + c$ **Ans.**

Do yourself - 12 :

Solve the following differential equations :

(i) $y' + 3y = e^{3x} y^2$ (ii) $xdy - \{y + xy^3 (1 + \ell nx)\}dx = 0$ (iii) $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$

7. TRAJECTORIES :

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

The trajectory will be called **Orthogonal** if each trajectory cuts every member of given family at right angle.

Working rule for finding orthogonal trajectory

1. Form the differential equation of family of curves

2. Write $-\frac{1}{dy/dx}$ for $\frac{dy}{dx}$ or $-\frac{r^2 d\theta}{dr}$ for $\frac{dr}{d\theta}$ if differential equation is in the polar form.

3. Solve the new differential equation to get the equation of orthogonal trajectories.

Note: A family of curves is self-orthogonal if it is its own orthogonal family.

Illustration 26: Find the value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses $x^2 + 2y^2 - y = c$.

Solution : Differentiate both sides of $x^2 + 2y^2 - y = c$ w.r.t. x, We get

$$2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

or $2x + (4y - 1)\frac{dy}{dx} = 0$, is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to obtain the differential equation of the orthogonal trajectories, we get

$$2x + \frac{(1-4y)}{\frac{dy}{dx}} = 0 \Rightarrow \frac{dy}{dx} = \frac{4y-1}{2x}$$

$$\Rightarrow \int\frac{dy}{4y-1} = \int\frac{dx}{2x} \Rightarrow \frac{1}{4}\ell n\,(4y-1) = \frac{1}{2}\ell n\,x + \frac{1}{2}\ell n\,a,$$ where a is any constant.

$\Rightarrow \ell n(4y - 1) = 2\,\ell n\,x + 2\,\ell n\,a$ or , $4y - 1 = a^2x^2$

or, $y = \frac{1}{4}a^2x^2 + \frac{1}{4}$, is the required orthogonal trajectory, which is of the form $y = cx^2 + k$ where

$c = \frac{a^2}{4}$, $k = \frac{1}{4}$. **Ans.**

Illustration 27 : Prove that $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ are self orthogonal family of curves.

Solution :

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{...(i)}$$

Differentiating (i) with respect to x, we have

$$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0 \quad \text{...(ii)}$$

From (i) and (ii), we have to eliminate λ.

Now, (ii) gives

$$\lambda = \frac{-\left[b^2x + a^2y\frac{dy}{dx}\right]}{x + y\frac{dy}{dx}}$$

$$\Rightarrow a^2 + \lambda = \frac{(a^2 - b^2)x}{x + y(dy/dx)}, \quad b^2 + \lambda = \frac{-(a^2 - b^2)y(dy/dx)}{x + y(dy/dx)}$$

Substituting these values in (i), we get

$$\left(x + y\frac{dy}{dx}\right)\left(x - y\frac{dx}{dy}\right) = a^2 - b^2. \qquad \text{....(iii)}$$

as the differential equation of the given family.

Changing dy/dx to –dx/dy in (iii), we obtain

$$\left(x - y\frac{dx}{dy}\right)\left(x + y\frac{dy}{dx}\right) = a^2 - b^2. \qquad \text{...(iv)}$$

which is the same as (iii). Thus we see that the family (i) as self-orthogonal, i.e., every member of the family (i) cuts every other member of the same family orthogonally.

Do yourself - 13 :

(i) Find the orthogonal trajectories of the following families of curves :

(a) $x + 2y = C$ (b) $y = Ce^{-2x}$

Note :

Following exact differentials must be remembered :

(i) $xdy + y\,dx = d(xy)$

(ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{xdy + ydx}{xy} = d\left(\ell nxy\right)$

(v) $\dfrac{dx + dy}{x + y} = d\left(\ell n(x + y)\right)$

(vi) $\dfrac{xdy - ydx}{xy} = d\left(\ell n\dfrac{y}{x}\right)$

(vii) $\dfrac{ydx - xdy}{xy} = d\left(\ell n\dfrac{x}{y}\right)$

(viii) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right]$

(xi) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2y^2}$

(xii) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dx - e^x dy}{y^2}$

(xiii) $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^y dy - e^y dx}{x^2}$

Illustration 28 : Solve $\dfrac{y + \sin x\cos^2(xy)}{\cos^2(xy)}dx + \left(\dfrac{x}{\cos^2(xy)} + \sin y\right)dy = 0.$

Solution : The given differential equation can be written as;

$$\frac{y\,dx + x\,dy}{\cos^2(xy)} + \sin x\,dx + \sin y\,dy = 0.$$

$\Rightarrow$ $\sec^2(xy)\,d(xy) + \sin x\,dx + \sin y\,dy = 0$

$\Rightarrow$ $d(\tan(xy)) + d(-\cos x) + d(-\cos y) = 0$

$\Rightarrow$ $\tan(xy) - \cos x - \cos y = c.$ **Ans.**

Do yourself - 14 :

Solve the following differential equations :

(i) $x\,dx + y\,dy + 4y^3(x^2 + y^2)dy = 0.$ **(ii)** $x\,dy - ydx - (1 - x^2)dx = 0.$

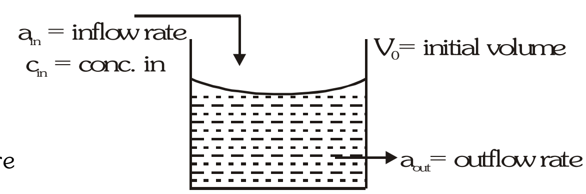
8. APPLICATION OF DIFFERENTIAL EQUATIONS :

(a) Mixing Problems

A chemical in a liquid solution with given concentration c_{in} gm/lit. (or dispersed in a gas) runs into a container with a rate of a_{in} lit/min. holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate (a_{out} litre/min.). In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula.

$$\begin{matrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs}\end{pmatrix} \qquad \text{.......(i)}$$

Arrival rate = (conc. in) (inflow rate) = c_{in} a_{in}



If y(t) denotes the amount of substance in the tank at time t & V(t) denotes the amount of mixture in tank at that time

$$\text{Departure rate} = \begin{pmatrix}\text{concentration in}\\ \text{container at time t}\end{pmatrix}.\ (\text{outflow rate})$$

$$= \frac{y(t)}{V(t)}\,.\,(a_{out})$$

where volume of mixture at time t, V(t) = initial volume + (inflow rate − outflow rate) t

$$= V_0 + (a_{in} - a_{out})t$$

Accordingly, Equation (i) becomes

$$\frac{dy(t)}{dt} = (\text{chemical's given arrival rate}) - \frac{y(t)}{V(t)}\cdot(\text{out flow rate}) \qquad \text{.......(ii)}$$

$$\frac{d(y(t))}{dt} = c_{in}a_{in} - \frac{y(t)}{V_0 + (a_{in} - a_{out})t}.a_{out}$$

This leads to a first order linear D.E. which can be solved to obtain y(t) i.e. amount of chemical at time 't'.

Illustration 29 : A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour ?

Solution : Let y(t) be the amount of salt after t min.

Given y(0) = 20 kg

rate in $=\left(\frac{0.03\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min.}}\right)=\frac{0.75\text{kg}}{\text{min.}}$

As $a_{in} = a_{out}$, so the tank always contains 5000 L of liquid so the conc. at time 't' is $\left(\frac{y(t)}{5000}\right)\frac{\text{kg}}{\text{L}}$

so rate out = $\left(\frac{y(t)}{5000}\frac{\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min}}\right)=\frac{y(t)}{200}\frac{\text{kg}}{\text{min}}$

$$\frac{dy(t)}{dt}=0.75-\frac{y(t)}{200}$$

by solving as linear D.E. or variable separable and using initial condition, we get

$y(t) = 150 - 130\ e^{-t/200}$

The amount of salt after 30 min is

$y(30) = 150 - 130\ e^{-30/100} = 38.1$ kg

Do yourself - 15 :

(i) A tank initially holds 10 lit. of fresh water. At t = 0, a brine solution containing $\frac{1}{2}$kg of salt per lit. is poured into the tank at a rate of 2 lit/min. while the well-stirred mixture leaves the tank at the same rate. Find

(a) the amount and

(b) the concentration of salt in the tank at any time t.

(b) **Exponential Growth and Decay :**

In general, if y(t) is the value of quantity y at time t and if the rate of change of y with respect to t is proportional to its value y(t) at that time, then

$$\frac{dy(t)}{dt} = ky(t), \quad \text{where k is a constant} \qquad ....(i)$$

$$\int\frac{dy(t)}{y(t)}=\int k\,dt$$

Solving, we get $y(t) = Ae^{kt}$

equation (i) is sometimes called the law of natural growth (if k > 0) or law of natural decay (if k < 0).

In the context of population growth, we can write

$$\frac{dP}{dt}=kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dt}=k$$

where k is growth rate divided by the population size; it is called **the relative growth rate.**

Illustration 30 : A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 kg of the material present and after two hours it is observed that the material has lost 10 percent of its original mass, find **(a)** an expression for the mass of the material remaining at any time t, **(b)** the mass of the material after four hours, and **(c)** the time at which the material has decayed to one half of its initial mass.

Solution : (a) Let N denote the amount of material present at time t.

So, $\frac{dN}{dt}-kN=0$

This differential equation is separable and linear, its solution is

$N = ce^{kt}$ ....(i)

At t=0, we are given that N = 50. Therefore, from (i), $50 = ce^{k(0)}$ or c = 50. Thus,

$N = 50e^{kt}$ ....(ii)

At t = 2, 10 percent of the original mass of 50kg or 5kg has decayed. Hence, at t = 2, N = 50 – 5 = 45. Substituting these values into (ii) and solving for k, we have

$45 = 50e^{2k}$ or $k = \frac{1}{2}\ell n\frac{45}{50}$

Substituting this value into (ii), we obtain the amount of mass present at any time t as

$N = 50e^{\frac{1}{2}(\ell n 0.9)t}$ .....(iii)

where t is measured in hours.

(b) We require N at t = 4. Substituting t = 4 into (iii) and then solving for N, we find $N = 50e^{-2\ \ell n\ (0.9)}$ kg

(c) We require when N = 50/2 = 25. Substituting N = 25 into (iii) and solving for t, we find

$$25 = 50e^{\frac{1}{2}(\ell n 0.9)t} \Rightarrow t = \ell n\left(\frac{1}{2}\right)\Big/\left[\frac{1}{2}\ell n(0.9)\right] \text{hours}$$

(c) **Temperature Problems :**

Newton's law of cooling, which is equally applicable to heating, states that the time rate of change of the temperature of body is proportional to the temperature difference between the body and its surrounding medium. Let T denote the temperature of the body and let T_m denote the temperature of the surrounding medium. Then the time rate of change in temperature of the body is $\frac{dT}{dt}$, and

Newton's law of cooling can be formulated as

$$\frac{dT}{dt} = -k(T - T_m), \text{ or as } \frac{dT}{dt} + kT = kT_m \quad \text{....(a)}$$

where k is a positive constant of proportionality. Once k is chosen positive, the minus sign is required in Newton's law to make $\frac{dT}{dt}$ negative in a cooling process, when T is greater than T_m and positive in a heating process, when T is less than T_m.

Illustration 31 : A metal bar at a temperature of 100 F is placed in a room at a constant temperature of 0 F. If after 20 minutes the temperature of the bar is 50 F, find **(a)** the time it will take the bar to reach the temperature of 25 F and **(b)** the temperature of the bar after 10 minutes.

Solution : Use equation (a) with $T_m = 0$; the surrounding medium here is the room which is being held at a constant temperature of 0 F. Thus we have

$$\frac{dT}{dt} + kT = 0$$

whose solution is $T = ce^{-kt}$ .....(i)

Since T = 100 F at t = 0 (the temperature of the bar is initially 100 F), it follows (i) that $100 = ce^{-k(0)}$ or 100 = c. Substituting this value into (i), we obtain $T = 100e^{-kt}$ .....(ii)

At t = 20, we are given that T = 50 F; hence from (ii),

$50 = 100e^{-20k}$ from which $k = -\frac{1}{20}\ell n\frac{50}{100}$

Substituting this value into (ii), we obtain the temperature of the bar at any time t as

$$T = 100e^{\left(\frac{1}{20}\ell n\frac{1}{2}\right)t} \text{ F} \qquad .....(iii)$$

(a) We require t when T = 25 F. Substituting T = 25 F into (iii), we have

$$25 = 100e^{\left(\frac{1}{20}\ell n\frac{1}{2}\right)t}$$

Solving, we find that t = 39.6 min.

(b) We require T when t = 10. Substituting t = 10 into (iii) and then solving for T, we find that

$$T = 100e^{\left(\frac{1}{20}\ell n\frac{1}{2}\right)\times 10} \text{ F}$$

It should be noted that since Newton's law is valid only for small temperature difference, the above calculations represent only a first approximation to the physical situation.

(d) Geometrical applications :

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at point P is $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$

(i) The equation of the tangent at P is $y - y_1 = \frac{dy}{dx}(x - x_1)$

x-intercept of the tangent = $x_1 - y_1\left(\frac{dx}{dy}\right)$

y-intercept of the tangent = $y_1 - x_1\left(\frac{dy}{dx}\right)$

(ii) The equation of normal at P is $y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$

x and y-intercepts of normal are ; $x_1 + y_1\frac{dy}{dx}$ and $y_1 + x_1\frac{dx}{dy}$

(iii) Length of tangent $= PT = |y_1|\sqrt{1 + (dx/dy)^2_{(x_1,\, y_1)}}$

(iv) Length of normal $= PN = |y_1|\sqrt{1 + (dy/dx)^2_{(x_1,\, y_1)}}$

(v) Length of sub-tangent = $ST = \left|y_1\left(\frac{dx}{dy}\right)_{(x_1,\, y_1)}\right|$

(vi) Length of sub-normal = $SN = \left|y_1\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}\right|$

(vii) Length of radius vector = $\sqrt{x_1^2 + y_1^2}$

Do yourself - 16 :

(i) At each point (x,y) of a curve the intercept of the tangent on the y-axis is equal to $2xy^2$. Find the curve.

(ii) Find the equation of the curve for which the normal at any point (x,y) passes through the origin.

Miscellaneous Illustrations :

Illustration 32 : Solve (y log x – 1) ydx = xdy.

Solution : The given differential equation can be written as

$$x\frac{dy}{dx}+y=y^2\log x \quad .....(i)$$

Divide by xy^2 . Hence $\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=\frac{1}{x}\log x$

Let $\frac{1}{y}=v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$ so that $\frac{dv}{dx}-\frac{1}{x}v=-\frac{1}{x}\log x \quad .....(ii)$

(ii) is the standard linear differential equation with $P=-\frac{1}{x}$, $Q=-\frac{1}{x}\log x$

I.F. $= e^{\int pdx} = e^{\int -1/x\,dx} = 1/x$

The solution is given by

$$v\cdot\frac{1}{x}=\int\frac{1}{x}\left(-\frac{1}{x}\log x\right)dx=-\int\frac{\log x}{x^2}dx=\frac{\log x}{x}-\int\frac{1}{x}\cdot\frac{1}{x}dx=\frac{\log x}{x}+\frac{1}{x}+c$$

$\Rightarrow \quad v = 1 + \log x + cx = \log ex + cx$

or $\quad \frac{1}{y}=\log ex+cx$ or y (log ex + cx) = 1. **Ans.**

Illustration 33: For a certain curve y = f(x) satisfying $\frac{d^2y}{dx^2}=6x-4$, f(x) has a local minimum value 5 when x = 1. Find the equation of the curve and also the global maximum and global minimum values of f(x) given that $0 \le x \le 2$.

Solution : Integrating $\frac{d^2y}{dx^2}=6x-4$, we get $\frac{dy}{dx}=3x^2-4x+A$

When x = 1, $\frac{dy}{dx}=0$, so that A = 1. Hence

$$\frac{dy}{dx}=3x^2-4x+1 \quad ...(i)$$

Integrating, we get $y = x^3 - 2x^2 + x + B$

When x = 1, y = 5, so that B = 5.

Thus we have $y = x^3 - 2x^2 + x + 5$.

From (i), we get the critical points x = 1/3, x = 1

At the critical point $x=\frac{1}{3}$, $\frac{d^2y}{dx^2}$ is negative.

Therefore at x = 1/3, y has a local maximum.

At x = 1, $\frac{d^2y}{dx^2}$ is positive.

Therefore at x = 1, y has a local minimum.

Also $f(1) = 5,\ f\left(\frac{1}{3}\right) = \frac{139}{27}$. $f(0) = 5,\ f(2) = 7$

Hence the global maximum value = 7, and the global minimum value = 5. **Ans.**

Illustration 34 : Solve $\frac{dy}{dx} = \tan y \cot x - \sec y \cos x$.

Solution : $\frac{dy}{dx} = \tan y \cot x - \sec y \cos x$.

Rearrange it :

$(\sin x - \sin y)\cos x\, dx + \sin x \cos y\, dy = 0$.

Put $u = \sin y$, So, $du = \cos y\, dy$:

Substituting, we get

$(\sin x - u)\cos x\, dx + \sin x\, du = 0$, $\frac{du}{dx} - u\frac{\cos x}{\sin x} = -\cos x$

The equation is first-order linear in u.

The integrating factor is

$$I = \exp\int -\frac{\cos x}{\sin x}dx = \exp\{-\ln(\sin x)\} = \frac{1}{\sin x}.$$

Hence, $$u\frac{1}{\sin x} = -\int\frac{\cos x}{\sin x}dx = -\ln\left|\sin x\right| + C.$$

Solve for u : $u = -\sin x \ln|\sin x| + C \sin x$.

Put y back : $\sin y = -\sin x \ln|\sin x| + C\sin x$. **Ans.**

Illustration 35 : Solve the equation $x\int_0^x y(t)dt = (x+1)\int_0^x t\,y(t)dt,\ x > 0$

Solution : Differentiating the equation with respect to x, we get

$$xy(x) + 1.\int_0^x y(t)dt = (x+1)xy(x) + 1.\int_0^x ty(t)dt$$

i.e., $$\int_0^x y(t)dt = x^2y(x) + \int_0^x ty(t)dt$$

Differentiating again with respect to x, we get $y(x) = x^2 y'(x) + 2xy(x) + xy(x)$

i.e., $(1 - 3x)y(x) = \frac{x^2 dy(x)}{dx}$

i.e., $\frac{(1-3x)dx}{x^2} = \frac{dy(x)}{y(x)}$, integrating we get

i.e., $y = \frac{c}{x^3}e^{-1/x}$ **Ans.**

Illustration 36 : **(Discontinuous forcing)** Solve : $y'+\frac{3}{x}y = g(x)$, where $g(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}$, and $y\left(\frac{1}{2}\right) = \frac{1}{8}$, and y(x) is continuous on $[0,\infty)$.

Solution : The idea is to solve the equation separately on $0 \le x \le 1$ and on $x > 1$, then match the pieces up at $x = 1$ to get a continuous solution.

$0 \le x \le 1 : y'+\frac{3}{x}y = 1$. The integrating factor is $I = \exp\int \frac{3}{x}dx = e^{3\ell nx} = x^3$.

Then $yx^3 = \int x^3 dx = \frac{1}{4}x^4 + C$.

The solution is $y = \frac{1}{4}x + \frac{C}{x^3}$

Plug in the initial condition $\frac{1}{8} = y\left(\frac{1}{2}\right) = \frac{1}{8} + 8C, C = 0$

The solution on the interval $0 \le x \le 1$ is $y = \frac{1}{4}x$.

Note that $y(1) = \frac{1}{4}$.

$x > 1$: $y' + \frac{3}{x}y = \frac{1}{x}$. The integrating factor is the same as before, so $yx^3 = \int x^2 dx = \frac{1}{3}x^3 + C$.

The solution is $y = \frac{1}{3} + \frac{C}{x^3}$.

In order, to get value of C, set $y(1) = \frac{1}{4}$

$$\frac{1}{4} = y(1) = \frac{1}{3} + C, \ C = -\frac{1}{12}$$

The solution on the interval $x > 1$ is $y = \frac{1}{3} - \frac{1}{12}\frac{1}{x^3}$

The complete solution is $y = \begin{cases} \frac{1}{4}x & \text{if } 0 \le x \le 1 \\ \frac{1}{3} - \frac{1}{12x^3} & \text{if } x > 1 \end{cases}$ **Ans.**

Illustration 37 : Let $y = f(x)$ be a differentiable function $\forall x \in R$ and satisfies :

$f(x) = x + \int_0^1 x^2 z f(z)dz + \int_0^1 x z^2 f(z) dz$. Determine the function.

Sol. We have , $f(x) = x + x^2\int_0^1 z f(z) dz + x\int_0^1 z^2 f(z)dz$

Let $f(x) = x + x^2\lambda_1 + x\lambda_2$

Now $\lambda_1 = \int_0^1 z f(z)dz = \int_0^1 ((1+\lambda_2)z + z^2\lambda_1)z\,dz = \frac{1+\lambda_2}{3} + \frac{\lambda_1}{4}$

$\Rightarrow \quad 9\lambda_1 - 4\lambda_2 = 4$ .....(i)

also $\lambda_2 = \int_0^1 z^2 f(z)dz = \int_0^1 ((1+\lambda_2)z^3 + z^4\lambda_1)\,dz = \frac{(1+\lambda_2)}{4} + \frac{\lambda_1}{5}$

$\Rightarrow \quad 15\lambda_2 - 4\lambda_1 = 5$ ....(ii) from (i) and (ii);

$$\lambda_1 = \frac{80}{119} \text{ and } \lambda_2 = \frac{61}{119}$$

$$\Rightarrow \quad f(x) = x + \frac{80}{119}x^2 + \frac{61}{119}x = \frac{20x}{119}(4+9x)$$

ANSWERS FOR DO YOURSELF

1 : (i) one, two (ii) two, two (iii) one, one

2 : (i) $y' - y = 2(1 - x)$ (ii) $x^2y'' + 2xy' - 2y = 0$ (iii) $y''' = 4y'$

3 : (i) $\ell ny^2 = x + \ell n|x| + k$ (ii) $-\frac{1}{2y^2} = \frac{1}{4}\sqrt{1+4x^2} + k$ (iii) $\sec y = -2\cos x + C$

4 : (i) $\frac{y-4x+2}{y-4x-2} = ce^{-4x}$ (ii) $2(x - y) = c + \sin 2(x + y)$

5 : (i) $x + 2y + \ell n|2x - y| + c = 0$ (ii) $\frac{y^2}{2} + 3y - \frac{3x^2}{2} + 5xy = 0$

6 : (i) $x = cye^{xy}$ (ii) $y = cx^2e^{-1/xy}$

7 : (i) $\ell n(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right) + c$ (ii) $\sqrt{x^2 - y^2} = \sin^{-1}\left(\frac{y}{x}\right) + c$

8 : (i) 3 (ii) 2/3 (iii) (a) homogeneous (b) homogeneous (c) not homogeneous

9 : (i) $(3x + y)(x - y) = c_0$ (ii) $y\ell n\left(\frac{y}{x}\right) - y + x\ell nx + cx = 0$ (iii) $x^2y(2x + y) = 3$

10 : (i) $x + y - 3 = C(x - y + 1)^3$

11 : (i) $y = \frac{x^4}{2} + 6x^2\ell n|x| - 2x + cx^2$ (ii) $y = 2(x-a)^3 + \frac{c}{(x-a)^3}$ (iii) $2x\,\ell n\, y = \ell n^2 y + C$.

12 : (i) $y = \frac{1}{(c-x)e^{3x}}$ (ii) $\frac{x^2}{y^2} = -\frac{2}{3}x^3\left(\frac{2}{3} + \ell nx\right) + c$ (iii) $\frac{1}{y} = -\sin x + ce^x$.

13 : (i) (a) $y - 2x = K$ (b) $y^2 = x + K$

14 : (i) $\frac{1}{2}\ell n(x^2 + y^2) + y^4 = C$ (ii) $y + x^2 + 1 = Cx$

15 : (i) (a) $-5e^{-0.2t} + 5$ kg (b) $\frac{1}{2}(-e^{0.2t} + 1)$ kg/ℓ

16 : (i) $\frac{x}{y} = x^2 + C$ (ii) $x^2 + y^2 = C$

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. The order and degree of the differential equation $\left(1+3\frac{dy}{dx}\right)^{\frac{2}{3}}=4\frac{d^3y}{dx^3}$ are -

(A) $1,\ \frac{2}{3}$ (B) 3 , 1 (C) 1, 2 (D) 3, 3

2. The degree and order of the differential equation of the family of all parabolas whose axis is x-axis are respectively

(A) 2 , 1 (B) 1 , 2 (C) 3 , 2 (D) 2 , 3

3. The order and degree of the differential equation $\sqrt[3]{\frac{dy}{dx}}-4\frac{d^2y}{dx^2}-7x=0$ are a and b, then a + b is -

(A) 3 (B) 4 (C) 5 (D) 6

4. The order of the differential equation whose general solution is given by $y=(C_1+C_2)\cos(x+C_3)-C_4e^{x+c_5}$ where C_1, C_2, C_3, C_4, C_5 are arbitrary constants, is - **[JEE 98]**

(A) 5 (B) 4 (C) 3 (D) 2

5. The differential equation of the family of curves represented by $y = a + bx + ce^{-x}$ (where a, b, c are arbitrary constants) is -

(A) $y''' = y'$ (B) $y''' + y'' = 0$ (C) $y''' - y'' + y' = 0$ (D) $y''' + y'' - y' = 0$

6. The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant is -

(A) $(x^2 - y^2)\ y' = 2xy$ (B) $2(x^2 + y^2)\ y' = xy$

(C) $2(x^2-y^2)\ y' = xy$ (D) $(x^2 + y^2)y' = 2xy$

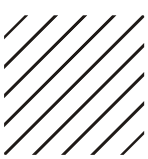
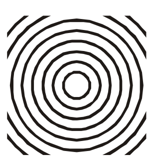
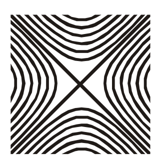
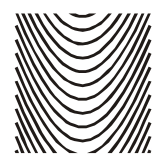
7. Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation $D^3y - 3D^2y - 4Dy + 12y = 0$ is -

(A) 0 (B) 1 (C) 2 (D) more than 2

8. If $y = e^{(K+1)x}$ is a solution of differential equation $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$, then k =

(A) –1 (B) 0 (C) 1 (D) 2

9. The general solution of the differential equation $\frac{dy}{dx}=\frac{1-x}{y}$ is a family of curves which looks most like which of the following ?

(A)  (B)  (C)  (D) 

10. The solution to the differential equation $y\ell ny + xy' = 0$, where $y(1) = e$, is -

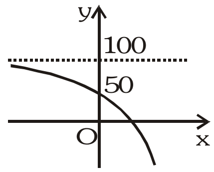
(A) $x(\ell ny) = 1$ (B) $xy(\ell ny) = 1$ (C) $(\ell ny)^2 = 2$ (D) $\ell ny+\left(\frac{x^2}{2}\right)y = 1$

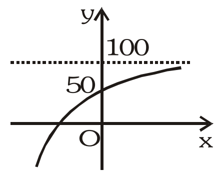
11. The equation of the curve passing through origin and satisfying the differential equation $\frac{dy}{dx} = \sin(10x + 6y)$ is -

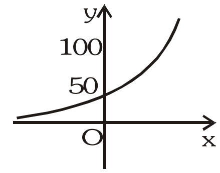
(A) $y=\frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right)-\frac{5x}{3}$ (B) $y=\frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4+3\tan 4x}\right)-\frac{5x}{3}$

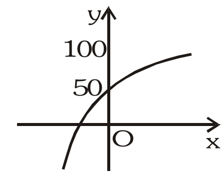
(C) $y=\frac{1}{3}\tan^{-1}\left(\frac{3+\tan 4x}{4-3\tan 4x}\right)-\frac{5x}{3}$ (D) none of these

12. Which one of the following curves represents the solution of the initial value problem Dy = 100 – y, where y(0) = 50

(A) 

(B) 

(C) 

(D) 

13. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x ty(t)dt = x^2y(x), (x > 0)$ is -

(A) $x^2 + y^2 = 13$ (B) $y^2 = \frac{9}{2}x$ (C) $\frac{x^2}{8} + \frac{y^2}{18} = 1$ (D) xy = 6

14. A curve passes through the point $\left(1, \frac{\pi}{4}\right)$ & its slope at any point is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$. Then the curve has the equation -

(A) $y = x\tan^{-1}\left(\ell n \frac{e}{x}\right)$ (B) $y = x\tan^{-1}(\ell n + 2)$ (C) $y = \frac{1}{x}\tan^{-1}\left(\ell n \frac{e}{x}\right)$ (D) none

15. The solution of the differential equation $(2x - 10y^3)\frac{dy}{dx} + y = 0$ is -

(A) $x + y = ce^{2x}$ (B) $y^2 = 2x^3 + c$ (C) $xy^2 = 2y^5 + c$ (D) $x(y^2 + xy) = 0$

16. Solution of differential equation $\left(1 + y^2\right)dx + \left(x - e^{\tan^{-1}y}\right)dy = 0$ is -

(A) $y\,e^{\tan^{-1}x} = \tan^{-1}x + c$ (B) $x\,e^{\tan^{-1}y} = \frac{1}{2}e^{2\tan^{-1}y} + c$

(C) $2x = e^{\tan^{-1}y} + c$ (D) $y = x\,e^{-\tan^{-1}x} + c$

17. The general solution of the differential equation, $y' + y\phi'(x) - \phi(x) \cdot \phi'(x) = 0$ where $\phi(x)$ is a known function is -

(A) $y = ce^{-\phi(x)} + \phi(x) - 1$ (B) $y = ce^{\phi(x)} + \phi(x) + K$ (C) $y = ce^{-\phi(x)} - \phi(x) + 1$ (D) $y = ce^{-\phi(x)} + \phi(x) + K$

18. The solution of the differential equation, $e^x(x + 1)dx + (ye^y - xe^x)dy = 0$ with initial condition f(0) = 0, is -

(A) $xe^x + 2y^2e^y = 0$ (B) $2xe^x + y^2e^y = 0$ (C) $xe^x - 2y^2e^y = 0$ (D) $2xe^x - y^2e^y = 0$

19. The solution of the differential equation $ydx + (x + x^2y)dy = 0$ is -

(A) $\frac{1}{xy} + \log y = c$ (B) log y = cx (C) $-\frac{1}{xy} = c$ (D) $-\frac{1}{xy} + \log y = c$

20. The solution of $y^5x + y - x\frac{dy}{dx} = 0$ is -

(A) $x^4/4 + 1/5\,(x/y)^5 = C$ (B) $x^5/5 + (1/4)\,(x/y)^4 = C$

(C) $(x/y)^5 + x^4/4 = C$ (D) $(xy)^4 + x^5/5 = C$

21. The solution of $\frac{xdy}{x^2 + y^2} = \left(\frac{y}{x^2 + y^2} - 1\right)dx$ is -

(A) y = x cot (c – x) (B) $\cos^{-1} y/x = -x + c$

(C) y = x tan (c – x) (D) $y^2/x^2 = x\tan(c - x)$

<u>SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)</u>

22. The value of the constant 'm' and 'c' for which y = mx + c is a solution of the differential equation $D^2y - 3Dy - 4y = -4x$

(A) is m = –1 (B) is c = 3/4 (C) is m = 1 (D) is c = –3/4

23. If $x\frac{dy}{dx}$ = y (log y – log x + 1), then the solution of the equation is -

(A) $\log\left(\frac{x}{y}\right) = cy$ (B) $\log\left(\frac{y}{x}\right) = cx$ (C) $y = xe^{cx}$ (D) $x = ye^{cx}$

24. Solutions of the differential equation $x^2\left(\frac{dy}{dx}\right)^2 + xy\left(\frac{dy}{dx}\right) - 6y^2 = 0$ -

(A) $y = cx^2$ (B) $x^3 y = c$ (C) $xy^3 = c$ (D) $y = cx$

25. A solution of the differential equation, $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is - [JEE 99]

(A) y = 2 (B) y = 2x (C) y = 2x – 4 (D) $y = 2x^2 - 4$

26. The solution the differential equation $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}\left(e^x + e^{-x}\right) + 1 = 0$ is are -

(A) $y + e^{-x} = c$ (B) $y - e^{-x} = c$ (C) $y + e^x = c$ (D) $y - e^x = c$

27. The solution of $\frac{dy}{dx} = \frac{ax+h}{by+k}$ represent a parabola if -

(A) a = –2, b = 0 (B) a = –2, b = 2 (C) a = 0, b = 2 (D) a = 0, b = 0

28. A normal is drawn at a point P(x, y) of a curve. It meets the x-axis and the y-axis in point A and B, respectively, such that $\frac{1}{OA} + \frac{1}{OB} = 1$, where O is the origin, the equation of such a curve is a circle which passes through (5, 4) and has -

(A) centre (1, 1) (B) centre (2, 1) (C) radius 5 (D) radius 4

CHECK YOUR GRASP — **ANSWER KEY** — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	B	C	C	B	A	C	C	B	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	A	B	D	A	C	B	A	B	D	B
Que.	21	22	23	24	25	26	27	28		
Ans.	C	C,D	B,C,D	A,B	C	A,D	A,C	A,C		

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. Which one of the following is homogeneous function ?

(A) $f(x, y) = \dfrac{x-y}{x^2+y^2}$

(B) $f(x, y) = x^{\frac{1}{3}}.y^{-\frac{2}{3}}\tan^{-1}\dfrac{x}{y}$

(C) $f(x, y) = x(\ell n\sqrt{x^2+y^2} - \ell n y) + ye^{x/y}$

(D) $f(x, y) = x\left[\ell n\dfrac{2x^2+y^2}{x} - \ell n(x+y)\right] + y^2\tan\dfrac{x+2y}{3x-y}$

2. The graph of the function y = f(x) passing through the point (0, 1) and satisfying the differential equation $\dfrac{dy}{dx} + y\cos x = \cos x$ is such that -

(A) it is a constant function

(B) it is periodic

(C) it is neither an even nor an odd function

(D) it is continuous & differentiable for all x.

3. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality k > 0 depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and k = 1/15 then the time to drain the tank if the water is 4 meter deep to start with is -

(A) 30 min (B) 45 min (C) 60 min (D) 80 min

4. The solution of the differential equation, $x^2\dfrac{dy}{dx}.\cos\dfrac{1}{x} - y\sin\dfrac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$ is -

(A) $y = \sin\dfrac{1}{x} - \cos\dfrac{1}{x}$

(B) $y = \dfrac{x+1}{x\sin\frac{1}{x}}$

(C) $y = \cos\dfrac{1}{x} + \sin\dfrac{1}{x}$

(D) $y = \dfrac{x+1}{x\cos\frac{1}{x}}$

5. If $y = \dfrac{x}{\ell n|cx|}$ (where c is an arbitrary constant) is the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x} + \phi\left(\dfrac{x}{y}\right)$ then the function $\phi\left(\dfrac{x}{y}\right)$ is -

(A) $\dfrac{x^2}{y^2}$ (B) $-\dfrac{x^2}{y^2}$ (C) $\dfrac{y^2}{x^2}$ (D) $-\dfrac{y^2}{x^2}$

6. If $\int_a^x ty(t)dt = x^2 + y(x)$ then y as a function of x is -

(A) $y = 2 - (2+a^2)e^{\frac{x^2-a^2}{2}}$

(B) $y = 1 - (2+a^2)e^{\frac{x^2-a^2}{2}}$

(C) $y = 2 - (1+a^2)e^{\frac{x^2-a^2}{2}}$

(D) none

7. A function f(x) satisfying $\int_0^1 f(tx)dt = nf(x)$, where x > 0, is -

(A) $f(x) = c.x^{\frac{1-n}{n}}$ (B) $f(x) = c.x^{\frac{n}{n-1}}$ (C) $f(x) = c.x^{\frac{1}{n}}$ (D) $f(x) = c.x^{(1-n)}$

8. The differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}+\sin y+x^2=0$ is of the following type -

(A) linear (B) homogeneous (C) order two (D) degree one

9. A curve C passes through origin and has the property that at each point (x, y) on it the normal line at that point passes through (1, 0). The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is -

(A) x = 0 (B) y = 0 (C) y = x + 1 (D) x + y + 1 = 0

10. The function f(x) satisfying the equation, $f^2(x) + 4f'(x) \cdot f(x) + [f'(x)]^2 = 0$ is -

(A) $f(x) = c \cdot e^{(2-\sqrt{3})x}$ (B) $f(x) = c \cdot e^{(2+\sqrt{3})x}$ (C) $f(x) = c \cdot e^{(\sqrt{3}-2)x}$ (D) $f(x) = c \cdot e^{-(2+\sqrt{3})x}$

11. The equation of the curve passing through (3, 4) & satisfying the differential equation,

$y\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-x=0$ can be -

(A) x − y + 1 = 0 (B) $x^2 + y^2 = 25$ (C) $x^2 + y^2 - 5x - 10 = 0$ (D) x + y − 7 = 0

12. Number of straight lines which satisfy the differential equation $\frac{dy}{dx}+x\left(\frac{dy}{dx}\right)^2-y=0$ is -

(A) 1 (B) 2 (C) 3 (D) 4

13. Let $y = (A + Bx)e^{3x}$ be a solution of the differential equation $\frac{d^2y}{dx^2}+m\frac{dy}{dx}+ny=0$, m, n ∈ I, then -

(A) m + n = 3 (B) $n^2 - m^2 = 64$ (C) m = −6 (D) n = 9

14. The differential equation $2xy\, dy = (x^2 + y^2 + 1)\, dx$ determines -

(A) A family of circles with centre on x-axis

(B) A family of circles with centre on y-axis

(C) A family of rectangular hyperbola with centre on x-axis

(D) A family of rectangular hyperbola with centre on y-axis

15. If $f''(x) + f'(x) + f^2(x) = x^2$ be the differential equation of a curve and let P be the point of maxima then number of tangents which can be drawn from point P to $x^2 - y^2 = a^2$, a ≠ 0 is -

(A) 2 (B) 1 (C) 0 (D) either 1 or 2

16. The solution of $x^2dy - y^2dx + xy^2(x - y)dy = 0$ is -

(A) $\ell n\left|\frac{x-y}{xy}\right|=\frac{y^2}{2}+c$ (B) $\ell n\left|\frac{xy}{x-y}\right|=\frac{x^2}{2}+c$ (C) $\ell n\left|\frac{x-y}{xy}\right|=\frac{x^2}{2}+c$ (D) $\ell n\left|\frac{x-y}{xy}\right|=x+c$

17. The orthogonal trajectories of the system of curves $\left(\frac{dy}{dx}\right)^2=\frac{4}{x}$ are -

(A) $9(y + c)^2 = x^3$ (B) $y + c = \frac{-x^{3/2}}{3}$ (C) $y + c = \frac{x^{3/2}}{3}$ (D) all of these

BRAIN TEASERS — ANSWER KEY — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A,B,C	A,B,D	C	A	D	A	A	C,D	A	C,D
Que.	11	12	13	14	15	16	17			
Ans.	A,B	B	A,C,D	C	A	A	A,B,C,D			

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

TRUE / FALSE

1. $f(x, y) = e^{y/x} + \ell nx - \ell ny$ is a homogeneous function of degree zero.
2. Consider the differential equation $y'' + 2y' + y = 0$. $y = e^{-t}$ is the solution of this differential equation but $y = te^{-t}$ is not the solution of differential equation.
3. The differential equation $y^3dy + (x + y^2)\, dx = 0$ becomes homogeneous if we put $y^2 = t$.
4. The degree of the differential equation $2\left(\frac{d^2y}{dx^2}\right)^2 + \frac{d^3y}{dx^3} - \frac{dy}{dx} - \sin^2 y + \sin\left(\frac{dy}{dx}\right) = 0$ is 2.
5. The differential equation of the family of parabola whose axis is parallel to y-axis has order 3 & degree 1.

MATCH THE COLUMN

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

1. Match the properties of the curves given in column-I with the corresponding curve(x) given in the column-II.

	Column-I		Column-II
(A)	A curve passing through (2, 3) having the property that length of the radius vector of any of its point P is equal to the length of the tangent drawn at this point, can be	(p)	Straight line
(B)	A curve passing through (1, 1) having the property that any tangent intersects the y-axis at the point which is equidistant from the point of tangency and the origin, can be	(q)	Circle
(C)	A curve passing through (1, 0) for which the length of normal is equal to the radius vector, can be	(r)	Parabola
(D)	A curve passes through the point (2, 1) and having the property that the segment of any of its tangent between the point of tangency and the x-axis is bisected by the y-axis, can be	(s)	Hyperbola

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : The order of the differential equation of all the circles which touches x-axis is 2.

because

Statement-II : The order of differential equation is same as number of independent arbitrary constant in the given curve.

(A) A (B) B (C) C (D) D

2. **Statement-I** : The order of the differential equation whose primitive is $y = A + \ell nBx$ is 2

because

Statement-II : If there are 'n' independent arbitrary constants in a family of curve then the order of the corresponding differential equation is 'n'.

(A) A (B) B (C) C (D) D

3. **Statement-I** : The orthogonal trajectory to the curve $(x - a)^2 + (y - b)^2 = r^2$ is $y = mx + b - am$ where a and b are fixed numbers and r & m are parameters.

because

Statement-II : In a plane, the line that passes through the centre of circle is normal to the circle.

(A) A (B) B (C) C (D) D

4. **Statement-I** : $\sin x \frac{d^2y}{dx^2} + \cos x \frac{dy}{dx} + \tan x = 0$ is not a linear differential equation.

because

Statement-II : A differential equation is said to be linear if dependent variable and its differential coefficients occurs in first degree and are not multiplied together.

(A) A (B) B (C) C (D) D

5. Consider the differential equation $(xy - 1)\frac{dy}{dx} + y^2 = 0$

Statement-I : The solution of the equation is $xy = \log y + c$.

because

Statement-II : The given differential equation can be expressed as $\frac{dx}{dy} + Px = Q$, whose integrating factor is ℓny.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

A & B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B . Both the reservoirs are filled completely with water , their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released , the quantity of water in reservoir A is 1.5 times the quantity of water in reservoir B.

Let V_A & V_B represents volume of reservoir A & B at any time t, then :

On the basis of above information, answer the following questions :

1. If after 1/2 an hour $V_A = kV_B$, then k is -

(A) 3 (B) 3/4 (C) $\sqrt{3}$ (D) none of these

2. After how many hours do both the reservoirs have the same quantity of water ?

(A) $\log_{4/3} 2$ hrs (B) $\log_{(4/3)} 4$ hrs (C) 2 hrs (D) $\frac{1}{2 - \log_2 3}$ hrs

3. If $\frac{V_A}{V_B} = f(t)$, where 't' is time. Then f(t) is -

(A) increasing (B) decreasing (C) non-monotonic (D) data insufficient.

Comprehension # 2

Let y = f(x) and y = g(x) be the pair of curves such that

(i) the tangents at point with equal abscissae intersect on y-axis.

(ii) the normals drawn at points with equal abscissae intersect on x-axis and

(iii) curve f(x) passes through (1, 1) and g(x) passes through (2, 3) then

On the basis of above information, answer the following questions :

1. The curve f(x) is given by -

(A) $\frac{2}{x} - x$ (B) $2x^2 - \frac{1}{x}$ (C) $\frac{2}{x^2} - x$ (D) none of these

2. The curve g(x) is given by -

(A) $x-\frac{1}{x}$ (B) $x+\frac{2}{x}$ (C) $x^2-\frac{1}{x^2}$ (D) none of these

3. The value of $\int_1^2 (g(x)-f(x))dx$ is -

(A) 2 (B) 3 (C) 4 (D) 4 $\ell n2$

MISCELLANEOUS TYPE QUESTION	ANSWER KEY	EXERCISE-3

- ***True / False***

  1. T 2. F 3. T 4. F 5. T

- ***Match the Column***

  1. (A)→(p,s), (B)→(q), (C)→(q,s), (D)→(r)

- ***Assertion & Reason***

  1. A 2. D 3. A 4. D 5. C

- ***Comprehension Based Questions***

  Comprehension # 1 : 1. C 2. A,D 3. B

  Comprehension # 2 : 1. A 2. B 3. B

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

1. State the order and degree of the following differential equations :

(a) $\left[\frac{d^2x}{dt^2}\right]^3 + \left[\frac{dx}{dt}\right]^4 - xt = 0$ (b) $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

2. Form a differential equation for the family of curves represented by ax + by = 1 , where a & b are arbitary constants.

3. Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g , f & c are arbitary constants.

4. Form the differential equation of circles passing through the points of intersection of unit circle with centre at the origin and the line bisecting the first quadrant.

5. Obtain the differential equation associated with the primitive , $y = c_1 e^{3x} + c_2 e^{2x} + c_3 e^x$, where c_1 , c_2 , c_3 are arbitrary constants.

6. Solve : $\frac{\ell n(\sec x + \tan x)}{\cos x} dx = \frac{\ell n(\sec y + \tan y)}{\cos y} dy$

7. Solve : $(1 - x)(1 - y)\, dx = xy(1 + y)\, dy$

8. Solve : $\frac{dy}{dx} + \frac{\sqrt{(x^2-1)(y^2-1)}}{xy} = 0$

9. Solve : $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

10. Solve : $\frac{dy}{dx} = \frac{x(2\ell n x + 1)}{\sin y + y\cos y}$

11. Solve : $\frac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 + x^2 - y^2}{x^2 - y^2}}$

12. Solve : $\frac{dy}{dx} + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$

13. Solve : $e^{(dy/dx)} = x + 1$ given that when x = 0 , y = 3

14. Solve : $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

15. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Determine the equation of the curve.

16. It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of time if at t = 0 , the mass of the radius was m_0 and during time t_0 α % of the original mass of radium decay.

17. Solve : $\sin x.\frac{dy}{dx} = y.\ \ell n y$ if y = e , when $x = \frac{\pi}{2}$

18. Find the curve y = f (x) where $f(x) \geq 0$, $f(0) = 0$, bounding a curvilinear trapezoid with the base [0, x] whose area is proportional to $(n + 1)^{th}$ power of f(x). It is known that $f(1) = 1$.

19. Solve : $\left[x\cos\frac{y}{x} + y\sin\frac{y}{x}\right] y = \left[y\sin\frac{y}{x} - x\cos\frac{y}{x}\right] x\frac{dy}{dx}$

20. Solve : $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$

21. Solve : $(x - y)\, dy = (x + y + 1)\, dx$

22. Solve : $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

23 Solve : $\frac{dy}{dx} = \frac{2(y + 2)^2}{(x + y - 1)^2}$

24. Solve : $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$

25. Solve : (a) $\frac{dy}{dx} = \frac{x^2 + xy}{x^2 + y^2}$ (b) $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2y)\, dy$

26. Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.

27. The light rays emanating from a point source situated at origin when reflected from the mirror of a search light are reflected as beam parallel to the x-axis. Show that the surface is parabolic, by first forming the differential equation and then solving it.

28. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1 , 1).

29. Find the curve for which the sum of the lengths of the tangent and subtangent at any of its point is proportional to the product of the co–ordinates of the point of tangency, the proportionality factor is equal to k.

30. Use the substitution $y^2 = a - x$ to reduce the equation $y^3.\frac{dy}{dx} + x + y^2 = 0$ to homogeneous form and hence solve it. (where a is variable)

Solve the following differential equations (Q. 31 to 45) :

31. $(x + \tan y)\, dy = \sin 2y\, dx$

32. $\frac{dy}{dx} + \frac{x}{1 + x^2}\, y = \frac{1}{2x(1 + x^2)}$

33. $(1 - x)\frac{dy}{dx} + 2xy = x(1 - x)^{1/2}$

34. $x(x - 1)\frac{dy}{dx} - (x - 2)\, y = x^3(2x - 1)$

35. $(1 + y + xy)dx + (x + x^3)dy = 0$

36. $y - x\, Dy = b(1 + x\, Dy)$

37. $\frac{dy}{dx} + \frac{y}{x}\ \ell ny = \frac{y}{x^2}(\ell n\, y)^2$

38. $\frac{dy}{dx} + xy = y\, e^{x/2}.\ \sin x$

39. $2\frac{dy}{dx} - y \sec x = y^3 \tan x$

40. $x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x$

41. $y\,(2xy + e^x)\, dx - e^x\, dy = 0$

42. $\sin x\frac{dy}{dx} + 3y = \cos x$

43. $x(x + 1)\frac{dy}{dx} = y\,(1 - x) + x^3.\ \ell nx$

44. $x\frac{dy}{dx} - y = 2x\ \operatorname{cosec} 2x$

45. $(1 + y)\, dx = (\tan^{-1} y - x)dy$

46. Find the curve for which the area of the triangle formed by the x–axis , the tangent line and radius vector of the point of tangency is equal to a^2.

Solve the following differential equations (Q. 47 to 56) :

47. $(x - y)\, dx + 2xy\, dy = 0$

48. $(x^3 + y^2 + 2)\, dx + 2y\, dy = 0$

49. $x\frac{dy}{dx} + y\ \ell ny = xye^x$

50. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)\, e^x \sec y$

51. $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$

52. $\left(\frac{dy}{dx}\right)^2 - (x+y)\frac{dy}{dx} + xy = 0$

53. $\frac{dy}{dx} = \frac{y^2 - x}{2y\,(x + 1)}$

54. $(1 - xy + x^2 y^2)\, dx = x^2\, dy$

55. $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

56. $yy' \sin x = \cos x\,(\sin x - y^2)$

57. Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point, $\sqrt{x^2 + y^2} = c\,e^{\pm \tan^{-1}\frac{y}{x}}$

58. A tank consists of 50 liters of fresh water. Two liters of brine each litre containing 5 gms of dissolved salt are run into tank per minute ; the mixture is kept uniform by stirring , and runs out at the rate of one litre per minute. If 'm' grams of salt are present in the tank after t minute, express 'm' in terms of t and find the amount of salt present after 10 minutes.

CONCEPTUAL SUBJECTIVE EXERCISE — **ANSWER KEY** — EXERCISE-4(A)

1. (a) order 2 & degree 3; (b) order 2 & degree 2

2. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

3. $[1 + (y')].y''' - 3y'(y'') = 0$

4. $(y' - 1)(x^2 + y^2 - 1) + 2(x + yy')(x - y) = 0$

5. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

6. $\ell n^2 (\sec x + \tan x) - \ell n^2 (\sec y + \tan y) = c$

7. $\ell n\, x\, (1 - y) = c - \frac{1}{2} y - 2y + \frac{1}{2} x$

8. $\sqrt{x^2 - 1} - \sec^{-1} x + \sqrt{y^2 - 1} = c$

9. $y = c\,(1 - ay)\,(x + a)$

10. $y \sin y = x\ \ \ell n\, x + c$

11. $\sqrt{x^2 - y^2} + \sqrt{1 + x^2 - y^2} = \frac{c\,(x + y)}{\sqrt{x^2 - y^2}}$

12. $\ell n\left|\tan\frac{y}{4}\right| = c - 2\sin\frac{x}{2}$

13. $y = (x + 1).\ \ell n\,(x + 1) - x + 3$

14. $\ell n\left[1 + \tan\frac{x + y}{2}\right] = x + c$

15. $x^2 + y^2 - 2x = 0,\ \ x = 1$

16. $m = m_0\, e^{-kt}$ where $k = -\frac{1}{t_0}\ \ell n\left(1 - \frac{\alpha}{100}\right)$

17. $y = e^{\tan(x/2)}$

18. $y = x^{1/n}$

19. $xy \cos\frac{y}{x} = c$

20. $\tan^{-1}\frac{y + 3}{x + 2} + \ell n\, c\sqrt{(y + 3)^2 + (x + 2)^2} = 0$

21. $\arctan\frac{2y + 1}{2x + 1} = \ell n\, c\sqrt{x^2 + y^2 + x + y + \frac{1}{2}}$

22. $(x+y-2) = c\,(y - x)^3$

23 $e^{-2\tan^{-1}\frac{y + 2}{x - 3}} = c.\ (y+2)$

24. $x+y+\frac{4}{3} = ce^{3(x-2y)}$

25. (a) $c(x - y)^{2/3}\,(x + xy + y)^{1/6} = \exp\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x + 2y}{x\sqrt{3}}\right]$ where $\exp x \equiv e^x$ (b) $y - x = c\,(y + x)$

26. $\frac{y^2 + y\sqrt{y^2 - x^2}}{x^2} = \ell n\left|y + \sqrt{y^2 - x^2}\,\frac{c^2}{x^3}\right|$, where same sign has to be taken.

28. $x + y - 2x = 0$

29. $y = \frac{1}{k}\,\ell n\left|c\left(k^2x^2 - 1\right)\right|$

30. $\frac{1}{2}\,\ell n\left|x^2 + a^2\right| - \tan^{-1}\left(\frac{a}{x}\right) = c$, where $a = x + y^2$

31. $x\sqrt{\cot y} = c + \sqrt{\tan y}$

32. $y\sqrt{1 + x^2} = c + \frac{1}{2}\,\ell n\left[\tan\frac{1}{2}\,\text{arc}\tan x\right]$ other form is $y\sqrt{1 + x^2} = c + \frac{1}{2}\,\ell n\frac{\sqrt{1 + x^2} - 1}{x}$

33. $y = c\,(1 - x) + \sqrt{1 - x^2}$

34. $y\,(x - 1) = x^2\,(x^2 - x + c)$

35. $xy = c - \text{arc}\tan x$

36. $y(1 + bx) = b + cx$

37. $x = \ell n y\left(cx^2 + \frac{1}{2}\right)$

38. $e^{-x/2} = y\,(c + \cos x)$

39. $\frac{1}{y^2} = -1 + (c + x)\cot\left(\frac{x}{2} + \frac{\pi}{4}\right)$

40. $x^3\, y^{-3} = 3\sin x + c$

41. $y^{-1}\, e^x = c - x$

42. $\left(\frac{1}{3} + y\right)\tan^3\frac{x}{2} = c + 2\tan\frac{x}{2} - x$

43. $4\,(x + 1)\,y + x^3\,(1 - 2\,\ell n x) = cx$

44. $y = cx + x\ \ell n \tan x$

45. $x = ce^{-\text{arctany}} + \text{arc}\tan y - 1$

46. $x = cy \pm \frac{a^2}{y}$

47. $y + x\ \ell n\, ax = 0$

48. $y = 3x - 6x - x^3 + ce^{-x} + 4$

49. $x\ell n y = e^x(x - 1) + c$

50. $\sin y = (e^x + c)\,(1 + x)$

51. $cx + 2xe^{-y} = 1$

52. $y = ce^x$; $y = c + \frac{x^2}{2}$

53. $y^2 = -1 + (x + 1)\,\ell n\frac{c}{x+1}$ or $x + (x + 1)\,\ell n\frac{c}{x+1}$

54. $y = \frac{1}{x}\tan\left(\ell n\left|cx\right|\right)$

55. $e^y = c.\ \exp(-e^x) + e^x - 1$

56. $y^2 = \frac{2}{3}\sin x + \frac{c}{\sin^2 x}$

58. $m = 5t\left(1 + \frac{50}{50 + t}\right)$ gms ; $91\frac{2}{3}$ gms

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. Consider the differential equation $\frac{dy}{dx}+P(x)y=Q(x)$

(a) If two particular solutions of given equation u(x) and v(x) are known, find the general solution of the same equation in term of u(x) and v(x).

(b) If α and β are constants such that the linear combinations $\alpha.u(x)+\beta.v(x)$ is a solution of the given equation, find the relation between α and β.

(c) If w(x) is the third particular solution different from u(x) and v(x) then find the ratio $\frac{v(x)-u(x)}{w(x)-u(x)}$.

2. Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x – axis lies on the parabola $2y^2 = x$.

3. Solve : $\frac{dy}{dx} = y + \int_0^1 y\,dx$ given y = 1 , where x = 0

4. Solve : $x^3\frac{dy}{dx} = y^3 + y^2\sqrt{y^2-x^2}$

5. Find the integral curve of the differential equation , $x(1 - x\,\ell n\,y)\,.\frac{dy}{dx} + y = 0$ which passes through $\left(1, \frac{1}{e}\right)$.

6. Let the function ℓnf(x) is defined where f(x) exists for $x \ge 2$ & k is fixed positive real number, prove that if $\frac{d}{dx}(x.f(x)) \le -kf(x)$ then $f(x) \le A\,x^{-1-k}$ where A is independent of x.

7. Find the differentiable function which satifies the equation $f(x) = -\int_0^x f(t)\tan t\,dt + \int_0^x \tan(t-x)\,dt$, where $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

8. Find all functions f(x) defined on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with real values and has a primitive F(x) such that

$f(x) + \cos x\,.\,F(x) = \frac{\sin 2x}{(1+\sin x)^2}$. Find f(x).

9. A tank contains 100 litres of fresh water. A solution containing 1 gm/litre of soluble lawn fertilizer runs into the tank at the rate of 1 lit/min and the mixture in pumped out of the tank at the rate of 3 litres/min. Find the time when the amount of fertilizer in the tank is maximum.

10. Given two curves y = f(x), where f(x) > 0, passing through the points (0, 1) & $y = \int_{-\infty}^{x} f(t)\,dt$ passing through the points (0, 1/2). The tangents drawn to both curves at the points with equal abscissas intersect on the x-axis. Find the curve f(x).

11. Find the orthogonal trajectories for the given family of curves when 'a' is the parameter.

(a) $y = ax^2$ (b) $\cos y = ae^{-x}$

12. Let $f(x, y, c_1) = 0$ and $f(x, y, c_2) = 0$ define two integral curves of a homogeneous first order differential equation. If P_1 and P_2 are respectively the points of intersection of these curves with an arbitrary line, y = mx then prove that the slopes of these two curves at P_1 and P_2 are equal.

13. If y_1 & y_2 be solutions of the differential equation $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x alone, and $y_2 = y_1 z$, then prove that $z = 1 + a\,e^{-\int \frac{Q}{y_1}dx}$, 'a' being an arbitrary constant.

BRAIN STORMING SUBJECTIVE EXERCISE **ANSWER KEY** EXERCISE-4(B)

1. (a) y = u(x) + K(u(x) – v (x)) where K is any constant ; (b) $\alpha+\beta = 1$; (c) constant **2.** $y = 2x + 1 - e^{2x}$

3. $y = \frac{1}{3-e}(2e^x - e + 1)$ **4.** $xy = c(y + \sqrt{y^2 - x^2})$ **5.** $x(ey + \ell ny + 1) = 1$ **7.** $\cos x - 1$

8. $f(x) = -\frac{2\cos x}{(1+\sin x)^2} - Ce^{-\sin x}.\cos x$ **9.** $27\frac{7}{9}$ minutes **10.** $f(x) = e^{2x}$ **11.** (a) $x^2 + 2y^2 = c$, (b) $\sin y = ce^{-x}$

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. The solution of the differential equation $(x^2 - y^2)dx + 2xy\, dy = 0$ is- **[AIEEE-2002]**

(1) $x^2 + y^2 = cx$ (2) $x^2 - y^2 + cx = 0$ (3) $x^2 + 2xy = y^2 + cx$ (4) $x^2 + y^2 = 2xy + cx^2$

2. The differential equation, which represents the family of plane curves $y = e^{cx}$, is- **[AIEEE-2002]**

(1) $y' = cy$ (2) $xy' - \log y = 0$ (3) $x\log y = yy'$ (4) $y\log y = xy'$

3. The equation of the curve through the point (1, 0), whose slope is $\frac{y-1}{x^2+x}$ is- **[AIEEE-2002]**

(1) $(y - 1)(x + 1) + 2x = 0$ (2) $2x(y - 1) + x + 1 = 0$

(3) $x(y - 1)(x + 1) + 2 = 0$ (4) $x(y + 1) + y(x + 1) = 0$

4. The degree and order of the differential equation of the family of all parabolas whose axis is x-axis, are respectively- **[AIEEE-2003]**

(1) 2, 3 (2) 2, 1 (3) 1, 2 (4) 3, 2

5. The solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$, is - **[AIEEE-2003]**

(1) $xe^{2\tan^{-1}y} = e^{\tan^{-1}y} + k$ (2) $(x - 2) = ke^{-\tan^{-1}y}$

(3) $2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + k$ (4) $xe^{\tan^{-1}y} = \tan^{-1}y + k$

6. The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant is- **[AIEEE-2004]**

(1) $2(x^2 - y^2)y' = xy$ (2) $2(x^2 + y^2)y' = xy$ (3) $(x^2 - y^2)y' = 2xy$ (4) $(x^2 + y^2)y' = 2xy$

7. The solution of the differential equation $ydx + (x + x^2y)dy = 0$ is- **[AIEEE-2004]**

(1) $-\frac{1}{xy} = C$ (2) $-\frac{1}{xy} + \log y = C$ (3) $\frac{1}{xy} + \log y = C$ (4) $\log y = Cx$

8. The differential representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where $c > 0$, is a parameter, is of order and degree as follows- **[AIEEE-2005, IIT-1999]**

(1) order 1, degree 2 (2) order 1, degree 1 (3) order 1, degree 3 (4) order 2, degree 2

9. If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is- **[AIEEE-2005]**

(1) $y\log\left(\frac{x}{y}\right) = cx$ (2) $x\log\left(\frac{y}{x}\right) = cy$ (3) $\log\left(\frac{y}{x}\right) = cx$ (4) $\log\left(\frac{x}{y}\right) = cy$

10. The differential equation whose solution is $Ax^2 + By^2 = 1$, where A and B are arbitrary constants is of- **[AIEEE-2006]**

(1) first order and second degree (2) first order and first degree

(3) second order and first degree (4) second order and second degree

11. The differential equation of all circles passing through the origin and having their centres on the x-axis is- **[AIEEE-2007]**

(1) $x^2 = y^2 + xy\frac{dy}{dx}$ (2) $x^2 = y^2 + 3xy\frac{dy}{dx}$ (3) $y^2 = x^2 + 2xy\frac{dy}{dx}$ (4) $y^2 = x^2 - 2xy\frac{dy}{dx}$

12. The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ satisfying the condition $y(1) = 1$ is- **[AIEEE-2008]**

(1) $y = \ell nx + x$ (2) $y = x\ell nx + x^2$ (3) $y = xe^{(x-1)}$ (4) $y = x\ell nx + x$

13. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is- **[AIEEE-2008]**

(1) $(x - 2)y'^2 = 25 - (y - 2)^2$ (2) $(y - 2)y'^2 = 25 - (y - 2)^2$

(3) $(y - 2)^2y'^2 = 25 - (y - 2)^2$ (4) $(x - 2)^2y'^2 = 25 - (y - 2)^2$

14. The differential equation which represents the family of curves $y = c_1 e^{c_2 x}$, where c_1 and c_2 are arbitrary constants, is :- **[AIEEE-2009]**

(1) yy" = y' (2) yy" = $(y')^2$ (3) y' = y^2 (4) y" = y'y

15. Solution of the differential equation cos x dy = y(sin x – y)dx, $0 < x < \frac{\pi}{2}$ is - **[AIEEE-2010]**

(1) sec x = (tan x + c) y (2) y sec x = tan x + c

(3) y tan x = sec x + c (4) tan x = (sec x + c) y

16. If $\frac{dy}{dx} = y + 3 > 0$ and y(0) = 2, then y(ln 2) is equal to :- **[AIEEE-2011]**

(1) 13 (2) –2 (3) 7 (4) 5

17. Let I be the purchase value of an equipment and V(t) be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where k > 0 is a constant and T is the total life in years of the equipment. Then the scrap value V(T) of the equipment is :- **[AIEEE-2011]**

(1) $I - \frac{k(T-t)^2}{2}$ (2) e^{-kT} (3) $T^2 - \frac{I}{k}$ (4) $I - \frac{kT^2}{2}$

18. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact, is given by : **[AIEEE-2011]**

(1) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 2$ (2) 2y – 3x = 0 (3) $y = \frac{6}{x}$ (4) $x^2 + y^2 = 13$

19. Consider the differential equation $y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$. It y(1) =1, then x is given by : **[AIEEE-2011]**

(1) $1 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (2) $4 - \frac{2}{y} - \frac{e^{\frac{1}{y}}}{e}$ (3) $3 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (4) $1 + \frac{1}{y} - \frac{e^{\frac{1}{y}}}{e}$

20. The population p(t) at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5\,p(t) - 450$. If p(0) = 850, then the time at which the population becomes zero is : **[AIEEE-2012]**

(1) ln18 (2) 2 ln18 (3) ln9 (4) $\frac{1}{2}$ ln18

21. At present a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is : **[JEE (Main)-2013]**

(1) 2500 (2) 3000 (3) 3500 (4) 4500

PREVIOUS YEARS QUESTIONS — **ANSWER KEY** — EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans	1	4	1	3	3	3	2	3	3	3	3	4	3	2	1
Que.	16	17	18	19	20	21									
Ans	3	4	3	4	2	3									

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. A country has a food deficit of 10% . Its population grows continuously at a rate of 3% per year . Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after 'n' years, where 'n' is the smallest integer bigger than or equal to, $\frac{\ell n10 - \ell n9}{\ell n(1.04) - 0.03}$. **[JEE 2000 (Mains) 10M out of 200]**

2. **(a)** Let f(x), $x \geq 0$, be a nonnegative continuous function, and let $F(x) = \int_0^x f(t)dt,\ x \geq 0$. If for some $c > 0$, $f(x) \leq cF(x)$ for all $x \geq 0$, then show that f(x) = 0 for all $x \geq 0$.

   (b) A hemispherical tank of radius 2 meters is initially full of water and has an outlet of 12 cm^2 cross sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law V(t) =0.6 $\sqrt{2gh(t)}$, where V(t) and h(t) are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank. **[JEE 2001 (Mains) 5+10M out of 100]**

3. If y(t) is a solution of $(1 + t)\frac{dy}{dt} - ty = 1$ and y(0) = –1, then y(1) is equal to -

   (A) 1/2 (B) e + 1/2 (C) e – 1/2 (D) –1/2

   [JEE 2003, (Screening) 3M]

4. Let p(x) be a polynomial such that p(1) = 0 and $\frac{d}{dx}(p(x)) > p(x)$ for all $x \geq 1$ show that p(x) > 0, for all x > 1.

   [JEE 2003 (mains), 4M out of 60]

5. A conical flask of height H has pointed bottom and circular top of radius R. It is completely filled with a volatile liquid. The rate of evaporation of the liquid is proportional to the surface area of the liquid in contact with air, with the constant of proportionality K > 0. Neglecting the thickness of the flask, find the time it takes for the liquid to evaporate completely. **[JEE 2003 (mains), 4M out of 60]**

6. If y = y (x) and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x$, y(0) = 1, then $y\left(\frac{\pi}{2}\right)$ equals - **[JEE 2004, (Screening) 3M]**

   (A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) 1

7. A curve passes through (2, 0) and slope at point P (x, y) is $\frac{(x+1)^2 + (y-3)}{(x+1)}$. Find equation of curve and area between curve and x-axis in 4^{th} quadrant. **[JEE - 2004 (Mains) 4M out of 60]**

8. **(a)** The solution of primitive integral equation $(x^2 + y^2)\,dy = xy\,dx$, is y = y (x). If y (1) = 1 and $y(x_0) = e$, then x_0 is - **[JEE 2005, (Screening) 3+3M]**

   (A) $\sqrt{2(e^2-1)}$ (B) $\sqrt{2(e^2+1)}$ (C) $\sqrt{3}e$ (D) none of these

   (b) For the primitive integral equation $ydx + y^2dy = x\,dy$; $x \in R$, y > 0, y = y (x), y (1) = 1, then y (–3) is -

   (A) 3 (B) 2 (C) 1 (D) 5

9. If length of tangent at any point on the curve y = f(x) intercepted between the point and the x-axis is of length 1. Find the equation of the curve. **[JEE 2005 (Mains) 4M out of 60]**

10. A tangent drawn to the curve y = f(x) at P(x, y) cuts the x-axis and y-axis at A and B respectively such that BP:AP = 3 : 1, given that f(1) = 1, then - **[JEE 2006 (5M, –2M) out of 184]**

    (A) equation of the curve is $x\frac{dy}{dx} - 3y = 0$ (B) normal at (1, 1) is x + 3y = 4

    (C) curve passes through (2, 1/8) (D) equation of the curve is $x\frac{dy}{dx} + 3y = 0$

11. **(a)** Let f(x) be differentiable on the interval $(0, \infty)$ such that f(1) = 1, and $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$, for each x > 0. Then f(x) is - **[JEE 2007 (3+3M)]**

(A) $\frac{1}{3x}+\frac{2x^2}{3}$ (B) $\frac{-1}{3x}+\frac{4x^2}{3}$ (C) $\frac{-1}{x}+\frac{2}{x^2}$ (D) $\frac{1}{x}$

(b) The differential equation $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$ determines family of circles with

(A) variable radii and a fixed centre at (0, 1)
(B) variable radii and a fixed centre at (0, –1)
(C) fixed radius 1 and variable centres along the x-axis.
(D) fixed radius 1 and varialble centres along the y-axis.

12. Let a solution y = y(x) of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2)=\frac{2}{\sqrt{3}}$.

Statement-1 : $y(x)=\sec\left(\sec^{-1}x-\frac{\pi}{6}\right)$ **[JEE 2008 (3M, –1M)]**

and

Statement-2 : y(x) is given by $\frac{1}{y}=\frac{2\sqrt{3}}{x}-\sqrt{1-\frac{1}{x^2}}$

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.
(C) Statement-1 is True, Statement-2 is False.
(D) Statement-1 is False, Statement-2 is True.

13. Match the statements/ expressions in **Column I** with the open intervals in **Column II.** **[JEE 2009, 8M]**

	Column I		Column II
(A)	Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2y'+y=0$	(P)	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
(B)	Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(Q)	$\left(0,\frac{\pi}{2}\right)$
		(R)	$\left(\frac{\pi}{8},\frac{5\pi}{4}\right)$
(C)	Interval in which at least one of the points of local maximum of $\cos^2x+\sin x$ lies	(S)	$\left(0,\frac{\pi}{8}\right)$
(D)	Interval in which $\tan^{-1}(\sin x+\cos x)$ is increasing	(T)	$(-\pi, \pi)$

14. Let f be a real-valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x,y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(–3) is equal to **[JEE 10, 3M]**

15. **(a)** Let $f:[1,\infty)\to[2,\infty)$ be a differentiable function such that f(1) = 2. If $6\int_1^x f(t)dt=3x\,f(x)-x^3$ for all $x \geq 1$, then the value of $f(2)$ is **[JEE 2011, 4M]**

(b) Let $y'(x)+y(x)g'(x)=g(x)g'(x)$, y(0) = 0, $x\in R$, where $f'(x)$ denotes $\frac{d f(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is

[JEE 2011, 4M]

16. If y(x) satisfies the differential equation y' − ytanx = 2x sec x and y(0) = 0, then **[JEE 2012, 4M]**

(A) $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right)=\frac{\pi^2}{18}$ (C) $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right)=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$

17. Let $f:\left[\frac{1}{2},1\right]\to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right)=1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

[JEE(Advanced) 2013, 2M]

(A) (2e − 1, 2e) (B) (e − 1, 2e − 1)

(C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

18. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right), x>0$.

Then the equation of the curve is [JEE(Advanced) 2013, 2M]

(A) $\sin\left(\frac{y}{x}\right)=\log x+\frac{1}{2}$ (B) $\operatorname{cosec}\left(\frac{y}{x}\right)=\log x+2$

(C) $\sec\left(\frac{2y}{x}\right)=\log x+2$ (D) $\cos\left(\frac{2y}{x}\right)=\log x+\frac{1}{2}$

Paragraph for Question 55 and 56

Let $f : [0,1] \to \mathbb{R}$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0,1]$.

19. If the function $e^{-x}f(x)$ assumes its minimum in the interval [0,1] at $x=\frac{1}{4}$, which of the following is true?

[JEE(Advanced) 2013, 3, (−1)]

(A) $f'(x) < f(x)$, $\frac{1}{4}<x<\frac{3}{4}$ (B) $f'(x) > f(x)$, $0<x<\frac{1}{4}$

(C) $f'(x) < f(x)$, $0<x<\frac{1}{4}$ (D) $f'(x) < f(x)$, $\frac{3}{4}<x<1$

20. Which of the following is true for 0 < x < 1 ? **[JEE(Advanced) 2013, 3, (−1)]**

(A) $0 < f(x) < \infty$ (B) $-\frac{1}{2}<f(x)<\frac{1}{2}$ (C) $-\frac{1}{4}<f(x)<1$ (D) $-\infty<f(x)<0$

PREVIOUS YEARS QUESTIONS — **ANSWER KEY** — EXERCISE-5 [B]

2. (b) $\frac{7\pi x10^5}{135\sqrt{g}}$ sec **3.** D **5.** $\frac{H}{K}$ **6.** A **7.** (x − 3) (x + 1) = y − 3 ; $\frac{4}{3}$ units

8. (a) C (b) A **9.** $\sqrt{1-y^2}+ln\left|\frac{1-\sqrt{1-y^2}}{y}\right|=\pm x+c$ **10.** A,B,C, D **11.** (a) A (b) C

12. C **13.** (A) → (P, Q, S) ; (B) → (P, T) ; (C) → (P, Q, R, T) ; (D) → (S) **14.** 9

15. (a) Bonus; (b) 0 **16.** A,D **17.** D **18.** A **19.** C **20.** D